1. The parents of a teenager child woke up in the middle of the night and suspected their kid had broken their 11:00 p.m. curfew. The parents noticed that the child had forgotten a cup of coffee outside the front door. Since they were familiar with Newton's law of cooling, they thought they could use it to determine the time the child got home, and so they decided to measure its temperature. The temperature outside was 30°F and had not changed since 6:00 p.m. of the previous day. At 1:00 a.m. the temperature of the coffee was 35°F and at 2:00 a.m. its temperature was 32.5°F. The parents estimated that the temperature of the coffee must have been about 70 degrees (cold enough not to be noticed) when the child got home. At what time did the child get home?

   A. 9:00 p.m.

   B. 9:30 p.m.

   C. 10:00 p.m.

   D. 11:00 p.m.

   E. Midnight

1. The differential equation

$$\frac{dx}{dt} = \frac{1}{10}x(100 - x) - h$$

models a logistic population with harvesting at rate h. Determine the range of the values of h such that if the initial population is large enough, the population stabilizes, and does not go extinct for large t.

A. $h < 25$

B. $h < 100$

C. $h < 150$

D. $h < 250$

E. $h < 180$

1. The differential equation

$$\frac{dx}{dt} = \frac{1}{10}x(100 - x) - hx$$

models a logistic population with harvesting at rate hx. Determine the range of h such that the population stabilizes, and does not go extinct for large t, and determine the maximum of hx.

A. $h < 2$, maximum of hx is 2

B. $h < 2$, maximum of hx is 1

C. $h < 1$, maximum of hx is $\frac{1}{2}$

D. $h < 1$, maximum of hx is $\frac{1}{4}$

E. $h < 1$, maximum of hx is $\frac{1}{3}$

8. Find an implicit solution of the initial value problem:

$$\begin{cases} (6x^2y^2 + 4e^x - 2y\sin 2x) + (4x^3y + \cos 2x)\dfrac{dy}{dx} = 0, \\ y(0) = 1 \end{cases}.$$

A. $2x^3y^2 - y\cos 2x + 4e^x = 3$

B. $x^3y^2 + y\cos 2x + 4ye^x = 0$

C. $2x^3y^2 + y\cos 2x + 4e^x = 5$

D. $x^3y^2 + y^2\cos 2x + 4e^x = 5$

E. $2x^3y^2 + y\cos 2x - 4ye^x = -3$

3. The general solution of $xy' - y = x^2e^x$ is

A. $y = xe^x + cx$

B. $y = x^2e^x - xe^x + cx$

C. $y = xe^x - cx^2$

D. $y = x^2e^x + xe^x + cx$

E. None of the above

6. The solution of

$$y' + \frac{y}{x} = \frac{2}{x^2 y}, \qquad x \neq 0$$

is given by

A. $x^2y^2 + 4xy = C$

B. $x^2y^2 + 4x = C$

C. $xy^2 - 2x = C$

D. $x^2y^2 - 4x = C$

E. $xy^2 - 4x = C$

3. Find the explicit solution of the initial value problem

$$y' = \frac{xy^2}{x^2+1} \quad , \quad y(0) = 3.$$

A. $\frac{1}{2}(6 + \ln(1 + x^2))$

B. $\dfrac{6}{2 - 3\ln(1 + x^2)}$

C. $\dfrac{6}{2 + 3\ln(1 + x^2)}$

D. $\frac{1}{2}(6 - 3\ln(1 + x^2))$

E. $\frac{1}{3}(9 - 2\ln(1 + x^2))$

8. Which of the following is the general solution to $y'' + 4y = e^{2t} + 12\sin(2t)$?

A. $y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{8}e^{2t} - 3t\cos(2t)$

B. $y(t) = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4}t\,e^{2t} - 3t\cos(2t)$

C. $y(t) = c_1 + c_2 e^{-4t} + \frac{1}{12}te^{2t} - 3t\cos(2t)$

D. $y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{8}e^{2t} + 3\sin(2t)$

E. None of the above.

10. A particular solution of the equation

$$y^{(4)} - y'' = 2\sin t - 3e^{-t} + 4t$$

is of the form

A. $y_p(t) = A\cos t + B\sin t + Cte^{-t} + t^2(Dt + E)$

B. $y_p(t) = t(A\cos t + B\sin t) + Cte^{-t} + t^2(Dt + E)$

C. $y_p(t) = A\cos t + B\sin t + Ce^{-t} + t^2(Dt + E)$

D. $y_p(t) = t(A\cos t + B\sin t) + Ce^{-t} + t^2(Dt + E)$

E. $y_p(t) = t(A\cos t + B\sin t) + Cte^{-t} + t(Dt + E)$

9. According to the method of undetermined coefficients, what is the proper form of a particular solution Y to the following differential equation?

$$y^{(4)} - 4y'' = 24t^2 - 4 - 3te^t.$$

A. $Y(t) = At^2 + Bte^t$.

B. $Y(t) = At^2 + Bt + C + Dte^t + Ee^t$.

C. $Y(t) = At^3 + Bt^2 + Ct + D + Ete^t + Fe^t$.

D. $Y(t) = At^4 + Bt^3 + Ct^2 + Dte^t + Ee^t$.

E. None of the above.

5. How many asymptotically unstable equilibrium solution(s) does the following differential equation have?

$$y' = (y^2 + 1)(y^2 - 1)(y + 2).$$

A. 0,

B. 1,

C. 2,

D. 3,

E. None of the above.

3. Find the solution $y(x)$ to

$$\frac{dy}{dx} = e^{-\frac{y}{x}} + \frac{y}{x},$$
$$y(e) = 0.$$

A. $y(x) = \ln(\ln(x))$

B. $y(x) = x\ln(\ln(x))$

C. $y(x) = \ln(\ln(x) + 1) - \ln(2)$

D. $y(x) = x\ln(x) - e$

E. $y(x) = x\ln(x) - x$

8. Which of the following is the general solution to $y'' + 4y = e^{2t} + 12\sin(2t)$?

A. $y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{8}e^{2t} - 3t\cos(2t)$

B. $y(t) = c_1e^{2t} + c_2e^{-2t} + \frac{1}{4}t\,e^{2t} - 3t\cos(2t)$

C. $y(t) = c_1 + c_2e^{-4t} + \frac{1}{12}\,te^{2t} - 3t\cos(2t)$

D. $y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{8}e^{2t} + 3\sin(2t)$

E. None of the above.

9. Find the general solutions to $y'' + 6y' + 10y = 0$

A. $c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t)$

B. $c_1e^{-3t}\cos(t) + c_2e^{-3t}\sin(t)$

C. $c_1e^{-3t}\cos(2t) + c_2e^{-3t}\sin(2t)$

D. $c_1e^{2t}\cos(t) + c_2e^{2t}\sin(t)$

E. $c_1e^{3t}\cos(t) + c_2e^{3t}\sin(t)$

7. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

A. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \right],$

B. $c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{-t} \right],$

C. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \right],$

D. $c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \right],$

E. $c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \right].$

2. Find the number of **stable** critical points for the autonomous equation

$$\frac{dx}{dt} = x(x-1)^2(x+3)(x^2-4).$$

A. 1

B. 2

C. 3

D. 4

E. 0

16. One eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ -1 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is $\lambda = 3$. A basis for the corresponding eigenspace is

A. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

17. If $y(x)$ is a solution of $y'' - 2y' + y = 0$ satisfying $y(0) = 1$ and $y'(0) = -1$, then $y(\frac{1}{2}) =$

A. 0

B. $-e^{\frac{1}{2}}$

C. $e^{\frac{1}{2}}$

D. $2e^{\frac{1}{2}}$

E. $-3e^{\frac{1}{2}}$

19. The general solution of the differential equation $y'' + 4y = -8\frac{1}{\sin x}$ is

A. $2(\cos 2x)\ln|\sin 2x| + 4x\sin 2x$

B. $2(\sin 2x)\ln|\sin 2x| + 4x\cos 2x.$

C. $2\ln|\cos 2x| + 4x\cos 2x$

D. $2x\cos 2x + 4\sin 2x$

E. None of the above

24. The general solution of $\mathbf{x}' = \begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix} \mathbf{x}$ has the form

A. $c_1 e^t \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix}$

B. $c_1 e^t \begin{bmatrix} \cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\sin t \\ \cos t - 2\sin t \end{bmatrix}$

C. $c_1 e^t \begin{bmatrix} 2\cos t + \sin t \\ -\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$

D. $c_1 e^t \begin{bmatrix} \cos t - 2\sin t \\ \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} 2\cos t + \sin t \\ -\cos t \end{bmatrix}$

E. None of the above.

7. If $y = u_1 y_1 + u_2 y_2$ where $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ is a particular solution of

$$y'' - 4y = 4 \tan x$$

then u_1 and u_2 are determined by

A. $u_1' = e^{-2x} \tan x \quad u_2' = -e^{2x} \tan x$

B. $u' = -e^{-2x} \sec^2 x \quad u_2' = e^{2x} \sec^2 x$

C. $u_1' = -2 \sin 2x \tan x \quad u_2' = 2 \cos 2x \tan x$

D. $u_1' = 2 \sin 2x \tan x \quad u_2' = -2 \cos 2x \tan x$

E. $u_1' = \tan x \quad u_2' = 0$

10. Find all values of a such that the following system of equations has exactly one solution.

$$\begin{cases} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (a^2 - 5)z = a \end{cases}$$

A. $a \neq 2$ and $a \neq -2$

B. $a = 2$ or $a = -2$

C. $a = -2$

D. $a = \pm\sqrt{5}$

E. $a \neq 0$

17. Determine all values of k such that the vectors $(1, -1, 0)$, $(1, 2, 2)$, $(0, 3, k)$ are a basis for $\mathbb{R}^3$.

A. $k = 1$

B. $k = 2$

C. $k \neq 2$

D. $k \neq 1$

E. $k \neq 3$

11. Let $y(x)$ be the solution of the initial value problem

$$y'' + y = x,$$
$$y(0) = 1, \quad y'(0) = 2.$$

Then $y(\pi)$ is equal to

A. $y(\pi) = \pi$

B. $y(\pi) = 2\pi$

C. $y(\pi) = \pi - 1$

D. $y(\pi) = 2\pi + 1$

E. $y(\pi) = \pi + 1$

12. Let $y(x)$ satisfy

$$y'' - y' = 2\sin(x),$$
$$y(0) = 3, \quad y'(0) = 0.$$

Then $y(\pi)$ is equal to:

A. $e^{\pi} + 10$

B. $e^{\pi} + 1$

C. $e^{\pi} + 2$

D. $e^{\pi} + 3$

E. e^{π}

12. For the system $\mathbf{x}' = \begin{bmatrix} 5 & 5 \\ -8 & -7 \end{bmatrix} \mathbf{x}$, the origin is

A. a saddle point

B. a proper node source

C. a center point

D. a spiral sink

E. a spiral source

15. The solution curves of the system

$$X'(t) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} X(t),$$

where $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, have the following structure near the origin $x_1 = x_2 = 0$:

A. A saddle point

B. A nodal sink

C. A nodal source

D. A center

E. parallel lines.

16. Let $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Find all the values of a so that the origin $x_1 = x_2 = 0$

$$X'(t) = \begin{bmatrix} a & -27 \\ 3 & 0 \end{bmatrix} X(t)$$

is a spiral source of the system, that is, the solution curves are spirals approaching infinity as $t \to \infty$?

A. $0 < a < 27$

B. $0 < a < 18$

C. $-18 < a < 18$

D. $0 > a > -18$.

E. $0 > a > -27$.

17. Let $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ satisfy

$$X'(t) = \begin{bmatrix} 1 & 6 \\ 1 & 2 \end{bmatrix} X(t), \quad X(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The limit $S = \lim_{t\to\infty} \frac{x_2'(t)}{x_1'(t)}$, which is the slope of the curve $X(t)$ as $t \to \infty$, is equal to

A. $S = -3$

B. $S = \frac{1}{2}$

C. $S = -\frac{1}{3}$

D. $S = -\frac{1}{2}$

E. $S = 3$

20. The matrix

$$A = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

satisfies $\text{Det}(A - \lambda I) = -(1+\lambda)^3$. If

$$X(t) = e^{-t}\left(\frac{t^2}{2}W_1 + tW_2 + W_3\right), W_1 \neq 0, \quad W_2 \neq 0, W_3 \neq 0$$

satisfies the equation $X'(t) = AX(t)$, the vector W_3 must satisfy the following condition:

A. $W_3 \neq a \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

B. $W_3 \neq a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

C. $W_3 \neq a \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

D. $W_3 \neq a \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

E. $W_3 \neq a \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$