Abel's theorem

A. Eremenko

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Abel's Theorem. *Let*

$$\sum_{n=0}^{\infty} a_n = A \tag{1}$$

be a convergent series with sum A. Define

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1. \tag{2}$$

Then

$$\lim_{x \to 1-} f(x) = A, \tag{3}$$

where x is real.

Condition that the series is convergent implies that the radius of convergence of the series in (2) is at least 1, so definition (2) makes sense.

Proof of the Theorem. Replacing a_0 by $a_0 - A$ we may assume wlog that $A = 0$. Consider the partial sums $s_k = a_0 + \ldots + a_n$; and notice the equality

$$f(z) = (1 - z) \sum_{n=0}^{\infty} s_n z^n. \tag{4}$$

Since the series (1) is convergent its partial sums are bounded, so the series in (4) has radius of convergence at least 1. So the terms in the RHS can be rearranged:

$$\begin{aligned} & s_0 + s_1 z + s_2 z^2 + \ldots - (z s_0 + z^2 s_1 + z^3 s_2 + \ldots \\ = \; & s_0 + (s_1 - s_0) z + (s_2 - s_1) z^2 + \ldots = a_0 + a_1 z + a_2 z^2 + \ldots . \end{aligned}$$

Now to prove that $f(x) \to 0$ for real $x \to 0-$, we choose arbitrary $\epsilon > 0$ and find an integer N such that

$$|s_n| \le \epsilon \quad \text{for} \quad n \ge N.$$

Then

$$\left|(1-z)\sum_{n=N}^{\infty} s_n z^n\right| \le \epsilon|1-z|\sum_{n=N}^{\infty}|z|^n = \epsilon\frac{|1-z|}{1-|z|}.$$

If z is on $[0,1)$, the last fraction equals 1, so the tail of the series has absolute value at most ϵ and this holds for all $z \in (0,1)$. Now for the part up to $N-1$ of our series we evidently have

$$(1-z)\sum_{n=0}^{N-1} s_n z^n \to 0 \quad \text{as} \quad z \to 1,$$

since for fixed N the sum is bounded. Therefore, $|f(z)| < 2\epsilon$ when $z \in (0,1)$ is sufficiently close to 1, and this proves the Theorem.

Remark. We did not fully use that z is positive in (3). It is sufficient that $|1-z|/(1-|z|)$ stays bounded as $z \to 1$, and this is called a *non-tangential limit*. This means that z stays in some sector of opening $< \pi$ with the vertex at 1 bisected by the interval $(0,1)$. Such a sector is sometimes called a *Stolz angle*.

Example of application. Find the sum

$$S = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - 1/2 + 1/3 - 1/4 + \ldots.$$

This series is convergent by the alternating series criterion. So we can apply Abel's theorem to the function

$$f(z) = \sum_{1}^{\infty}\frac{(-1)^{n-1}z^n}{n}.$$

This function can be found explicitly: just differentiate and obtain

$$f'(z) = \sum_{n=1}^{\infty}(-1)^{n-1}z^{n-1} = \sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{1+z},$$

So

$$f(z) = \operatorname{Log}(1+z),$$

where we used that $f(0) = 0$, to choose the correct branch. This function is continuous at $z = 1$, so we can plug $z = 1$ and obtain that $S = \operatorname{Log} 2$.

Another example. Find the sum of the Leibniz series:

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - 1/3 + 1/4 - 1/5 + \ldots.$$

Let

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}.$$

Then

$$f'(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n} = \frac{1}{1+z^2},$$

so, since $f(0) = 0$,

$$f(z) - \int_0^z f'(t)dt = \operatorname{Arctan} z,$$

and plugging $z = 1$ we obtain the answer $S = \pi/4$.

It is said that Leibnitz decided to quit being a lawyer and a diplomat in order to pursue mathematics because of this discovery.

Abel's theorem permits to prescribe sums to some divergent series, this is called the *summation in the sense of Abel.* If we have an analytic function f in the unit disk and the limit in (3) exists, then we call this limit *the sum of the series (2) in the sense of Abel.*

Abel's theorem ensures that this is indeed a generalization of convergence in the ordinary sense: a convergent series is Abel-summable and its sum in the sense of Abel is the same as its ordinary sum.

Example. The divergent series

$$\sum_{n=1}^{\infty} (-1)^{n-1} n = 1 - 2 + 3 - 4 + \ldots$$

has Abel sum 1/4. Indeed, since

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (-1)^{n-1} n z^n = z \sum_{n=1}^{\infty} (-1)^n n z^{n-1} = z \frac{d}{dz} \sum_{n=1}^{\infty} (-1)^{n-1} z^n \\ &= z \frac{d}{dz} \frac{z}{1+z} = \frac{z}{(1+z)^2}. \end{aligned}$$

Plugging $z = 1$ we obtain that the sum equals $1/4$.

Exercise. Show that Abel's sum of

$$\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \ldots = \frac{1}{2}.$$

From this example we see that grouping the terms does not preserve Abel's sum.

It may happen that the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

which is used in Abel's summation has a (single valued) *analytic continuation.* Then some values of z with $|z|$ greater than the radius of convergence can be unambigously plugged to it. This is called *Euler's summation.* For example, in Euler's sense:

$$\sum_{n=0}^{\infty}(-2)^n = \left(\sum_{n=0}^{\infty}(-2)^n z^n\right)_{z=1} = \left.\frac{1}{1+2z}\right|_{z=1} = \frac{1}{3},$$

and similarly

$$\sum_{n=0}^{\infty} 2^n = \left.\frac{1}{1-2n}\right|_{z=1} = -1.$$

(The radius of convergence of both series is $1/2$.)

This becomes ambiguous if the analytic continuation of our function is not single valued, thus in our first example, it is not clear how to plug for example $z = -2$ into $\mathrm{Log}\,(1+z)$.