# Age of Earth, 2

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In his paper, Kelvin gives long physical arguments justifying his flat Earth approximation.

Let us solve the heat equation in a ball $|x| < R$ in 3 space, with zero boundary condition and constant initial condition:

$$u_t = k\Delta u,$$

$$u(x, 0) = T,$$
$$u(x, t) = 0 \quad \text{for} \quad |x| = R.$$

The solution is spherically symmetric: it is a function of $r = |x|$ only. To separate the variables, set

$$u(x, t) = f(t)g(r),$$

then

$$f' = -\lambda k f, \tag{1}$$

and

$$rg'' + 2g' + \lambda rg = 0. \tag{2}$$

Here $\lambda > 0$ (the Earth cools!) and we have the boundary conditions $g(R) = 0$ and $g(0)$ is finite.

This can be reduced to Bessel's equation, but it is easier to apply the power series method directly.

Equation (2) has a regular singularity at 0, with exponents $0, -1$. The negative exponent has to be rejected, because $g(0)$ must be finite.

Thus

$$g(r) = \sum_0^\infty a_n r^n,$$

and substituting this to the equation we obtain

$$\sum_2 n(n-1)a_n r^n + 2\sum_1 n a_n r^n + \lambda \sum_2 a_{n-2} r^n = 0.$$

As there is only one term with $r$, we conclude that $a_1 = 0$. Then we obtain a recurrence:

$$(n^2+n)a_n = -\lambda a_{n-2}.$$

From the recurrence we obtain that $a_n = 0$ for all odd $n$. For even $n$, the recurrence is easily solved:

$$a_{2n} = (-1)^n \lambda^n (2n+1)!.$$

Thus

$$g(r) = \sum_0^\infty \frac{(-1)^n (r\sqrt{\lambda})^{2n}}{(2n+1)!} = \frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}}.$$

Using the boundary condition $g(R) = 0$ we obtain

$$\lambda_n = \frac{\pi^2 n^2}{R^2}, \quad n = 1, 2, 3 \ldots.$$

Therefore

$$u(x,t) = \frac{1}{r}\sum_1^\infty b_n \exp\left(-\frac{k\pi^2 n^2 t}{R^2}\right) \sin\frac{\pi n r}{R}.$$

Using the initial condition we obtain that

$$Tr = \sum_1^\infty b_n \sin\frac{\pi n r}{R}, \quad 0 < r < R,$$

so by Fourier formulas

$$b_n = \frac{2T}{R}\int_0^R r \sin\frac{\pi n r}{R} dr = \frac{2TR}{\pi n}(-1)^{n+1},$$

so

$$u(x,t) = \frac{2TR}{\pi r}\sum_1^\infty \frac{(-1)^{n+1}}{n} \exp\left(\frac{-k\pi^2 n^2 t}{R^2}\right) \sin\frac{\pi n r}{R},$$

and

$$u_r(R,t) = -\frac{2T}{R}\sum_1^\infty \exp\left(-\frac{k\pi^2 n^2 t}{R^2}\right).$$

We cannot just pass to the limit as $r \to \infty$ term by term, because the RHS becomes divergent when $R = \infty$.

To obtain an approximation when $R \to \infty$, we use the Poisson summation formula ("Fourier Transform", section 7):

$$\sum_{-\infty}^{\infty} e^{-an^2} = \sqrt{\frac{\pi}{a}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/a},$$

with $a = k\pi^2 t/R^2$. This gives

$$\lim_{R\to\infty} u_r(R,t) = -\frac{T}{\sqrt{\pi k t}},$$

as Kelvin obtained by assuming $R = \infty$ from the beginning.