AHLFORS' CONTRIBUTION TO THE THEORY OF MEROMORPHIC FUNCTIONS

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This article is an exposition for non-specialists of Ahlfors' work in the theory of meromorphic functions. When the domain is not specified we mean meromorphic functions in the complex plane $\mathbf{C}$.

The theory of meromorphic functions probably starts with Briot-Bouquet's book [17], where the terms "pole", "essential singularity" and "meromorphic" were introduced and what is known now as the Casorati–Weierstrass Theorem was stated for the first time. A major discovery was Picard's theorem (1879), which says that a meromorphic function omitting three values in the extended complex plane $\bar{\mathbf{C}}$ is constant. The modern theory of meromorphic functions started with attempts to give an "elementary proof" of this theorem. These attempts culminated in R. Nevanlinna's theory which was published first in 1925. Nevanlinna's books [45] and [46] were very influential and shaped much of research in function theory in this century. Nevanlinna theory, also known as value distribution theory, was considered one of the most important areas of research in 1930-40, so it is not surprising that Ahlfors started his career with work in this subject (besides he was Nevanlinna's student). There are two very good sources about this early work of Ahlfors: the first volume of his collected papers [1], where most of the papers have been supplied with Ahlfors' own commentaries, and Drasin's article [22]. I used both sources substantially in preparing this lecture, trying to minimize the intersection with Drasin's survey.

1. Type problem.

All Ahlfors' papers on the theory of meromorphic functions were written in the period 1929-1941, and they are unified by some general underlying ideas. The central problem is the problem of type. And the main method is the Length – Area Principle in a broad sense. We start with explanation of the type problem.

According to the Uniformization Theorem, every open simply connected abstract Riemann surface[1] R is conformally equivalent to the complex plane $\mathbf{C}$ or to the unit disk $\mathbf{U}$. In the first case R is said to have *parabolic type* and in the second case – *hyperbolic type*. Now assume that a Riemann surface is given in some explicit way. How do we recognize its type? This is the formulation of the type problem, which occupied a central place in function theory in 1930-50. Ahlfors in [1], p. 84, gives credit to A. Speiser [50] for this formulation.

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[1]Connected one-dimensional complex analytic manifold.

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One very natural explicit way of describing a Riemann surface is this: let R be an abstract surface[2] and let

$$\pi : R \to \bar{\mathbf{C}} \tag{1.1}$$

be a topologically holomorphic map, which means that every point $x_0 \in R$ has a neighborhood V and a local coordinate $z : V \to \mathbf{C}$, $z(x_0) = 0$ such that $f(x) = z^m(x)$ in V, where m is an integer. Points x_0 for which $|m| \neq 1$ are called critical points. There is a unique conformal structure on R which makes π meromorphic.

The intuitive image connected with (1.1) is the Riemann surface "spread out over the sphere", which is an awkward English equivalent of "Überlagerungsfläche". We think of R as a union of pieces of spheres (sheets) placed over the sphere $\bar{\mathbf{C}}$ such that π is the vertical projection. The sheets are pasted together respecting the projection π. This is how Riemann surfaces are usually introduced in elementary textbooks as Riemann surfaces of multi-valued functions.

In what follows we reserve the term "Riemann surface" for a pair (R, π) as in (1.1). We shall frequently use *conformal metrics* on a Riemann surface; these are Riemannian metrics compatible with the conformal structure. In local coordinates such metric is expressed as $ds = p(z)|dz|$, where p is a non-negative measurable function. In all situations we consider, p will be continuous and positive everywhere except an isolated set of points. A conformal metric can be always pulled back via a topologically holomorphic map. To a conformal metric corresponds an *area element* $d\rho = (i/2)p^2(z)dz \wedge d\bar{z} = p^2(x + iy)dxdy$. The simplest example is the spherical Riemannian metric. This is what one usually uses to measure the distance between Indianapolis and Haifa, for example.

The type problem in this setting is to find out from the geometric information about (1.1) whether R is conformally equivalent to $\mathbf{C}$ or to $\mathbf{U}$ (assuming that R is open and simply connected). For example Picard's theorem gives the following necessary condition of parabolic type: if R is parabolic then it covers all points of $\bar{\mathbf{C}}$ with at most two exceptions.

In [2] Ahlfors gives a sufficient condition for parabolic type. Assume that π has the following property: every curve in $\bar{\mathbf{C}}$ has a lifting to R. This means that the only singularities of the multiple valued analytic function π^{-1} are algebraic branch points (critical points of π). We choose a point $x_0 \in R$ and exhaust R with disks (with respect to the pullback of the spherical metric) of radius r centered at x_0. Let $n(r)$ be the number of critical points of π in the disk of radius r. If

$$\int^{\infty} \frac{r\,dr}{n(r)} = \infty \tag{1.2}$$

then R is of parabolic type. If for example $R = \mathbf{C}$ and π is an elliptic function then the pullback of the spherical metric differs from the Euclidean metric by a factor bounded from above and below, so $n(r)/r^2$ is bounded from above and below and we see that condition (1.2) gives the right order of growth for $n(r)$.

In [4] Ahlfors goes further and considers *arbitrary* conformal Riemannian metrics on R such that R is complete. Let $L(r)$ be the length of the circumference of the

[2]Two real-dimensional orientable triangulable topological manifold.

circle of radius r centered at z_0 with respect to such a metric. Then R has parabolic type *if and only if there exists a metric of the described type with the property*

$$\int^{\infty} \frac{dr}{L(r)} = \infty. \tag{1.3}$$

Here is the proof of (1.3), which demonstrates how the Length – Area principle works. We may identify R with a disk in $\mathbf{C}$, $R = \{z \in \mathbf{C} : |z| < r_0\}$, $r_0 \leq \infty$, Let Γ_t be the circumference of the disk of radius t, centered at 0 with respect to a conformal metric ds. Then

$$2\pi \leq \int_{\Gamma_t} \frac{|dz|}{|z|}$$

and by the Schwarz inequality

$$4\pi^2 \leq L(t) \int_{\Gamma_t} \frac{|dz|}{|z|^2},$$

$$4\pi^2 \int_{r'}^{r''} \frac{dt}{L(t)} \leq \int_{r'}^{r''} dt \int_{\Gamma_t} \frac{|dz|}{|z|^2}.$$

The last double integral is the area of swept by Γ_t as t varies from r' to r'', measured with respect to the conformal metric $|z|^{-1}|dz|$. If our Riemann surface is hyperbolic, that is, $r_0 < \infty$, this area stays bounded as $r'' \to r_0$ and the integral (1.3) converges for every conformal metric ds. If the surface is parabolic we just take $ds = |dz|$.

Of course (1.3) does not give any effective solution of the type problem, but almost all known criteria of parabolic type, including (1.2), where the spherical metric is used, can be obtained by a special choice of concrete metric on R. Two sources about later developments in the type problem are [56] and [59].

Assume now that the type of R is known, and let θ be a conformal map of the disk or plane $D = \{z : |z| < r_0\}$, $r_0 \leq \infty$ onto R. Then $f = \pi \circ \theta$ becomes a meromorphic function in D which is almost uniquely defined (up to a conformal automorphism of D). One may be interested in how the geometric properties of π are connected with the asymptotic behavior of f. By asymptotic behavior of f we mean for example how the number of solutions of equations $f(z) = a$ in the disk $\{z : |z| \leq r\}$ behaves when $r \to r_0$. This is the subject of the value distribution theory.

2. Value distribution theory.

If f is a rational function its value distribution is controlled by its degree d, which is the number of preimages of a generic point. The main tool of the value distribution theory of meromorphic functions of R. Nevanlinna is the characteristic function $T_f(r)$, which replaces the degree in the case when f is transcendental. We explain the version of the Nevanlinna theory which was found by Shimizu and Ahlfors independently of each other.

Let us denote by $n(r, a) = n_f(r, a)$ the number of solutions of the equation $f(z) = a$ in the disk $\{z : |z| \leq r\}$, counting multiplicity. Here $a \in \bar{\mathbf{C}}$. By the argument principle and the Cauchy–Riemann equations we have

$$n(r, a) - n(r, \infty) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f'}{f - a} dz = \frac{r}{2\pi} \frac{d}{dr} \int_{-\pi}^{\pi} \log |f(re^{i\theta}) - a| \, d\theta.$$

We divide by r and integrate with respect to r, assuming for a moment that $f(0) \neq a, \infty$ and using the notation[3]

$$N(r,a) = \int_0^r \frac{n(t,a)}{t} dt$$

to obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta}) - a| \, d\theta = \log |f(0) - a| + N(r,a) - N(r,\infty). \tag{2.1}$$

This is the Jensen formula.

Let $d\rho = p^2(u + iv) du dv$ be the area element of a conformal metric, normalized such that the area of $\bar{\mathbf{C}}$ equals 1. We integrate (2.1) with respect to $a \in \bar{\mathbf{C}}$ against $d\rho$ and obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} U(f(re^{i\theta})) \, d\theta = U(f(0)) + \int_{\bar{\mathbf{C}}} N(r,a) \, d\rho(a) - N(r,\infty), \tag{2.2}$$

where

$$U(w) = \int_{\bar{\mathbf{C}}} \log |w - a| \, d\rho(a). \tag{2.3}$$

If we choose $d\rho$ to be the normalized spherical area element, that is,

$$p(w) = \frac{1}{\sqrt{\pi}(1 + |w|^2)}$$

then changing the order of integration shows

$$T(r) := \int_{\bar{\mathbf{C}}} N(r,a) \, d\rho(a) = \int_0^r \frac{A(t) \, dt}{t}, \tag{2.4}$$

where

$$A(r) = \frac{1}{\pi} \int_{|z| \leq r} \frac{|f'|^2}{(1 + |f|^2)^2} dm, \text{ where } dm \text{ is the Euclidean area element in } \mathbf{C}.$$

The geometric interpretation of $A(r)$ is the area of the disk $|z| \leq r$ with respect to the pullback of the spherical metric, or in other words, the *average covering number* of the Riemann sphere by the restriction of f to the disk $|z| \leq r$. The function $T(r)$ defined in (2.4) is called *characteristic* of f. It is analogous to the degree of a rational function in the sense that it measures the number of preimages of a generic

[3] If $f(0) = a$ this has to be regularized in the following way:

$$N(r,a) = \int_0^r \{n(t,a) - n(0,a)\} t^{-1} dt + n(0,a) \log r.$$

point.[4] If f is rational we have $T_f(r) = \deg f \cdot \log r + O(1)$, and for transcendental f we always have $T(r)/\log r \to \infty$ as $r \to \infty$.

Now if we use the spherical area element $d\rho$ in (2.3), the integrals can be evaluated which gives $U(w) = \log\sqrt{1+|w|^2} = \log([w,\infty])^{-1}$, where $[,]$ stands for the *chordal* distance on the Riemann sphere. Thus the first term in (2.2) is

$$m(r,\infty) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{1}{[f(re^{i\theta}),\infty]}d\theta,$$

and in general we can define

$$m(r,a) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{1}{[f(re^{i\theta}),a]}d\theta, \quad a \in \bar{\mathbf{C}}.$$

This is called the *proximity function*[5]. It becomes large when f is close to a in the average on the circle $|z| = r$. It is important that the proximity function is non-negative. With these notations (2.2) can be rewritten as

$$N(r,\infty) + m(r,\infty) = T(r) + m(0,\infty).$$

Now we notice that $T(r)$ does not change if we replace f by $L \circ f$ where L is a rotation of the sphere, which is a conformal automorphism which preserves spherical distance. Thus we obtain

$$N(r,a) + m(r,a) = T(r) + m(0,a) = T(r) + O(1),\ r \to \infty,\ a \in \bar{\mathbf{C}}. \tag{2.5}$$

This is the First Main Theorem (FMT) of Nevanlinna in the form of Shimizu–Ahlfors. It implies by the way that

$$N(r,a) \leq T(r) + O(1),\ r \to \infty, \text{for every } a \in \bar{\mathbf{C}}, \tag{2.6}$$

because the proximity function is non-negative. When compared with (2.4) this shows that the points a for which $N(r,a)$ is substantially less then $T(r)$ should be exceptional. This was put in a very precise form by Ahlfors and Nevanlinna [46], who extended the earlier results by Valiron and Littlewood. We state only one theorem of this sort:

(2.7) *For every $\epsilon > 0$ we have $m(r,a) = O(T(r))^{1/2+\epsilon}$ for all $a \in \bar{\mathbf{C}}\backslash E$, where E is a set of zero logarithmic capacity.*

The description of the exceptional set in this theorem cannot be substantially improved without making any additional assumptions about f. Very subtle examples constructed by Hayman [35] show that

[4] In fact, $T(r)$ also has algebraic properties of degree: it is a *logarithmic height* in the field of meromorphic functions. The importance of this fact was fully recognized only recently [55, 40, 27].

[5] Nevanlinna's original definition of proximity function differs by an additive term which is bounded when $r \to \infty$.

(2.8) *given an F_σ set $E \subset \bar{\mathbf{C}}$ of zero logarithmic capacity there exists a meromorphic function f and a sequence $r_k \to \infty$ such that $m_f(r_k, a) \sim T_f(r_k)$ for all $a \in E$.*

On the other hand, a lot of work has been done to improve the exceptional set for various subclasses of meromorphic functions (see [22] for a survey of these results).

The functions $T_f(r), N_f(r,a)$ and $m_f(r,a)$ provide a convenient way to describe the asymptotic behavior of f. The characteristic $T(r)$ measures the growth of the number of solutions of $f(z) = a$ in the disks $|z| \leq r$. The First Main Theorem shows that $T(r)$ provides upper bound for this number *for all* $a \in \bar{\mathbf{C}}$ and (2.4) or (2.7) show that for typical a this upper bound is attained. It is natural to define the *order* of a meromorphic function f by

$$\lambda = \limsup_{r \to \infty} \frac{\log T(r)}{\log r}.$$

3. Simply connected parabolic Riemann surfaces.

There are few known sufficient conditions for parabolic type. The one mentioned in Section 1 requires that all paths can be lifted via π so it cannot be applied to Riemann surfaces (1.1) with "singularities". Let us define this notion precisely.

Let $D(a,r) \subset \bar{\mathbf{C}}$ be the disk with respect to the spherical metric of radius r centered at $a \in \bar{\mathbf{C}}$. We fix $a \in \bar{\mathbf{C}}$ and consider a function σ which to every $r > 0$ puts into correspondence a component $\sigma(r)$ of the preimage $\pi^{-1}(D(a,r))$ in such a way that $r_1 > r_2$ implies $\sigma(r_1) \subset \sigma(r_2)$. Now there are two possibilities:
a) $\cap_{r>0}\sigma(r) = \{\text{one point}\} \subset R$ or
b) $\cap_{r>0}\sigma(r) = \emptyset$.
In the case b) we say that σ defines a (transcendental) *singular point* over a of the Riemann surface (R, π). Thus all possible σ's are in one to one correspondence with all "points", ordinary and singular.

A singular point σ over a is called *logarithmic* if for some $r > 0$ the restriction $\pi|_{\sigma(r)}$ is a universal covering over $D(a,r)\backslash\{a\}$.

The following classification belongs to Iversen: a singular point σ over a is called *direct* if there exists $r > 0$ such that π omits the value a in $\sigma(r)$. If such r does not exist, σ is called indirect. So logarithmic singular points are direct.

Examples. Let $R = \mathbf{C}$. If $\pi = \exp$ there are two singular points: one over 0 and one over ∞. Both are direct (logarithmic). If $\pi(z) = \sin z/z$ there are two indirect singular points over 0 and two direct (logarithmic) singular points over infinity. If $\phi(z) = z \sin z$ then there is one direct singular point over infinity, which is not logarithmic.

Nevanlinna [46] proved that an open simply connected Riemann surface having no critical points and finitely many singular points is of parabolic type. This was extended by Elfving [25] to allow finitely many critical points. In fact, they obtained much more precise information about meromorphic functions associated with these surfaces:

(3.1) Theorem. *Let $\pi : R \to \bar{\mathbf{C}}$ be a simply connected Riemann surface, θ a conformal mapping of the disk $|z| < r_0 \leq \infty$ onto R, and $f = \pi \circ \theta$. Assume that π has only finitely many critical points and that (R, π) has a finite number $p \geq 2$ of singular points. Let $d(a)$ be the number of singular points over a. Then:*

(i) (R, π) *is of parabolic type, so that $r_0 = \infty$;*

(ii) $T_f(r) \sim cr^{p/2}$, $r \to \infty$, *where* $c > 0$ *is a constant, and*
(iii) $m_f(r,a) \sim (2cd(a)/p)r^{p/2}$, $r \to \infty$, $a \in \bar{\mathbf{C}}$.

The simplest example of the situation described by this theorem is $p = 2$ and no critical points, then f is an exponential. If $p = 3$ and there are no critical points, f can be expressed in terms of Airy functions, which satisfy the differential equation $y'' + zy = 0$. Nevanlinna's proof is based on asymptotic integration of certain linear differential equations with polynomial coefficients, which makes it impossible to extend the method to any substantially wider class of Riemann surfaces (at least nobody has ever succeeded in this). In the same journal issue where Nevanlinna's theorem (3.1) appeared, Ahlfors' [4] was published, which gave a completely different approach to the problem. It is Ahlfors' approach which was the base of all subsequent generalizations. Roughly speaking, the argument goes in the following way. One dissects R in (1.1) into pieces such that for each piece an explicit conformal map of a plane domain onto this piece can be found. Then one pastes these pieces together and obtains a homeomorphism of $\mathbf{C}$ onto R. The point is to perform this construction in such a way that the explicitly constructed homeomorphism $\theta_1 : \mathbf{C} \to R$ is as close to conformal as possible. If one finds a quasiconformal homeomorphism θ_1 this implies that R has parabolic type. One can go further and find θ_1 close to a conformal θ in the following sense:

$$\theta = \theta_1 \circ \phi, \quad \text{where} \quad \phi(z) \sim cz,\ z \to \infty.$$

Then one can derive the approximate formulas for $N(r,a)$ if θ_1 is described explicitly. (Ahlfors' argument was in fact more complicated, but this is what was eventually distilled from his paper). The distortion theorem used to show that θ is close to θ_1 is called Teichmüller–Wittich–Belinskii Theorem: *if* ϕ *is a quasiconformal homeomorphism of the plane and*

$$\int_{|z|\geq 1} k(z)\frac{dm}{|z|^2} < \infty, \quad \text{where} \quad k(z) = |\phi_{\bar{z}}/\phi_z|$$

then ϕ *is conformal at infinity, that is, the limit* $\lim_{z\to\infty} \phi(z)/z \neq 0$ *exists.*

Using the described approach more and more general classes of open simply connected Riemann surfaces were subsequently introduced, their type determined, and the value distribution of corresponding meromorphic functions studied. We mention the work of Künzi, Wittich, Schubart, Pöschl, Ullrich, Le Van Thiem, Huckemann, Hällström, and Goldberg. Most of the results are described in the books of Wittich [59] and Goldberg-Ostrovskii [31] and in Goldberg's paper [29]. For further development see [23] – the only paper on the subject written in English.

Let us assume now that (R,π) is a Riemann surface of parabolic type and see how the topological properties of π influence the asymptotic behavior of the meromorphic function f. The most famous result in this direction is the Denjoy–Carleman–Ahlfors Theorem [3].

(3.2) Theorem. *If a simply connected parabolic Riemann surface has* $p \geq 2$ *direct singular points, then the corresponding meromorphic function satisfies*

$$\liminf_{r\to\infty} r^{-p/2} T(r) > 0.$$

Thus the number of direct singularities is at most $\max\{1, 2\lambda\}$*, where* λ *is the order of the meromorphic function.*

The story of this theorem is explained in detail in M. Sodin's talk [49]. We add only few remarks. Heins [36] proved that

(3.3) *the set of projections of direct singular points of a parabolic Riemann surface is at most countable.*

On the other hand, simple examples show that the set of direct singular points itself may have the power of the continuum. One cannot say anything about the size of the set of projections of *all* singular points, even if the growth of T_f is restricted: for every $\lambda \geq 0$ there exist meromorphic functions of order λ such that every point in $\bar{\mathbf{C}}$ has a singular point over it [26]. Goldberg [30] generalized Ahlfors' theorem by including a certain subclass of indirect singular points, which are called K-points. The property of being a K-point depends only on the restriction of π to $\sigma(r)$ with arbitrarily small $r > 0$, but only for some narrow classes of Riemann surfaces is it known how to determine effectively whether a singular point is a K-point.

4. The Second Main Theorem of value distribution theory.

To formulate the main result of value distribution theory we denote by $n_1(r) = n_{1,f}(r)$ the number of critical points of meromorphic function f in the disk $|z| \leq r$, counting multiplicity. It is easy to check that

$$n_{1,f}(r) = n_{f'}(r, 0) + 2n_f(r, \infty) - n_{f'}(r, \infty). \tag{4.1}$$

Now we apply the averaging as above:

$$N_1(r) = N_{1,f}(r) := \int_0^r \frac{n_1(t)\,dt}{t}.$$

If 0 is a critical point, the same regularization as before has to be made. The Second Main Theorem (SMT) says that *for any finite set* $\{a_1, \ldots, a_q\} \subset \bar{\mathbf{C}}$ *we have*

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leq 2T(r) + S(r), \tag{4.2}$$

where $S(r) = S_f(r)$ *is a small "error term",* $S_f(r) = o(T(r))$ *when* $r \to \infty, r \notin E$*, where* $E \subset [0, \infty)$ *is a set of finite measure*[6]. The SMT may be regarded as a very precise way of saying that the term $m(r, a)$ in the FMT (2.5) is relatively small for most $a \in \bar{\mathbf{C}}$. It is instructive to rewrite (4.2) using (2.5) in the following form. Let $\bar{N}(r, a)$ be the averaged counting function of *different* solutions of $f(z) = a$, that is, this time we don't count multiplicity. Then $\sum_j N(r, a_j) \leq \sum_j \bar{N}(r, a_j) + N_1(r)$, and we obtain

$$\sum_{j=1}^{q} \bar{N}(r, a_j) \geq (q - 2)T(r) + S(r). \tag{4.3}$$

Now Picard's theorem is an immediate consequence: if three values a_1, a_2 and a_3 are omitted by a meromorphic function f, then $N_f(r, a_j) \equiv 0,\ 1 \leq j \leq 3$, so the left side of (4.3) is zero and we obtain $T_f(r) = S_f(r)$, which implies that f is constant. Here is a more refined

[6] In fact $S(r)$ has much better estimate. Recently there was a substantial activity in the study of the best possible estimate of this error term, see for example [37]. On the other hand Hayman's examples (2.8) show that in general the error term may not be $o(T(r))$ for all r, so an exceptional set E is really required.

(4.4) Corollary from the SMT. *Let $a_1, \dots, a_5$ be five points on the Riemann sphere. Then at least one of the equations $f(z) = a_j$ has simple solutions.*

Indeed, if all five equations have only multiple solutions then $N_1(r, f) \geq (1/2) \sum_{j=1}^{5} N(r, a_j)$. When we combine this inequality with SMT (4.2) it implies $(5/2)T(r) \leq 2T(r) + S(r)$, so $f = \text{const}$.

For most "reasonable" functions, like Nevanlinna's functions described in Theorem (3.1), the SMT tends to be an asymptotic equality rather then inequality; the most general class of functions for which this is known consists of meromorphic functions whose critical and singular points lie over a finite set. This is due to Teichmüller; see, for example [59], Ch. 4.

The purpose of Ahlfors' paper [7], as he explains in the commentary, "was to derive the main results of Nevanlinna theory of value distribution in the simplest way I knew how." The proof presented in [7] was the source of most generalizations of Nevanlinna theory to higher dimensions. We will return to these generalizations in Section 8 and now Ahlfors' proof will be presented. A reader not interested in proofs may skip to the end of this section.

Ahlfors' proof of SMT. Let $d\omega$ be the spherical area element, so that

$$d\omega = \frac{dudv}{\pi(1 + |w|^2)^2}, \quad w = u + iv.$$

We consider an area element on the Riemann sphere, $d\rho = p^2 d\omega$, where p is given by

$$\log p(w) := \sum_{j=1}^{q} \log \frac{1}{[w, a_j]} - 2 \log \left(\sum_{j=1}^{q} \log \frac{1}{[w, a_j]} \right) + C, \tag{4.5}$$

where $[,]$ is the chordal distance, and $C > 0$ is chosen so that

$$\int_{\bar{\mathbf{C}}} d\rho = 1.$$

(The sole purpose of the second term in the definition of p in (4.5) is to make this integral converge, without altering much the behavior near a_j which is determined by the first term).

We pull back this $d\rho$ via f and write the change of variable formula:

$$\int_{\bar{\mathbf{C}}} n(r, a) d\rho(a) = \int_0^r \int_{-\pi}^{\pi} p^2(w) \frac{|w'|^2}{(1 + |w|^2)^2} t \, d\theta dt, \quad w = f(te^{i\theta}). \tag{4.6}$$

Now we consider the derivative of the last double integral with respect to r, divided by $2\pi r$:

$$\lambda(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|w'|^2}{(1 + |w|^2)^2} p^2(w) \, d\theta, \quad w = f(re^{i\theta}).$$

Using the integral form of the arithmetic-geometric means inequality,[7] we obtain

$$\log \lambda(r) \geq \frac{1}{\pi} \int \log p(w) \, d\theta - \frac{1}{\pi} \int \log(1 + |w|^2) d\theta + \frac{1}{\pi} \int \log |w'| \, d\theta. \tag{4.7}$$

[7] $\frac{1}{b-a} \int_a^b \log g(x) \, dx \leq \log \left\{ \frac{1}{b-a} \int_a^b g(x) \, dx \right\}$.

The first integral in the right-hand side of (4.7) is approximately evaluated using (4.5) (the second summand in (4.5) becomes irrelevant because of another log); the second integral equals $4m(r,\infty)$ and the third one is evaluated using Jensen's formula (2.1):

$$(2+o(1))\sum_{j=1}^{q} m(r,a_j) + 2\left\{N(r,0,f') - N(r,\infty,f') - 2m(r,\infty)\right\} \leq \log\lambda(r)$$

The expression inside the brackets is equal to $N_1(r) - 2T(r)$ (by definition of N_1 and the FMT (2.5) applied with $a=\infty$), so

$$\sum_{j=1}^{q} m(r,a_j) + N_1(r) - (2+o(1))T(r) \leq \frac{1}{2}\log\lambda(r). \tag{4.8}$$

To estimate λ we return to the left side of (4.6). Integrating twice and using the FMT, we obtain $\int_0^r \frac{dt}{t}\int_0^t \lambda(s)sds = \int_{\overline{\mathbf{C}}} N(r,a)\,d\rho(a) \leq T(r) + O(1)$. Now the argument is concluded with the following elementary calculus lemma:
If g is an increasing function on $[0,\infty)$, tending to infinity, then $g'(x) \leq g^{1+\epsilon}(x)$ for all $x \notin E$, where E is a set of finite measure.
Applying this lemma twice, we conclude that $\log\lambda(r) = S(r)$, which proves the theorem.

5. Ahlfors' Überlagerungsflächentheorie.

In a series of papers written in 1932-33 Ahlfors developed a different approach to value distribution theory, which is based on a generalization of the Riemann–Hurwitz formula.

To explain his motivation we return to the type problem. As was explained in Section 1, Picard's theorem may be considered as a necessary condition of parabolic type. But this condition is very unstable: a small perturbation of (R,π) destroys the property that a value is omitted. The same can be said about (4.4) which may also be regarded as a necessary condition for parabolic type. This instability was noticed by A. Bloch [14], who stated the following general Continuity Principle, which I prefer to cite from Ahlfors' survey [5] *une proposition de nature qualitative, exacte avec un certain énoncé, demeure encore exacte si l'on modifie les données de la proposition, en leur faisant subir une déformation continue.* In accordance with this principle Bloch conjectured among other things the following:

(5.1) *let $D_1,\ldots,D_5$ be five Jordan domains with disjoint closures, and f be a non-constant meromorphic function in $\mathbf{C}$. Then there is a bounded simply connected domain $D \subset \mathbf{C}$ which is mapped by f homeomorphically onto one of the domains D_j.*

This conjecture improves (4.4) in accordance with the Continuity Principle. A similar improvement of Picard's theorem would be that

given three disjoint domains and a non-constant meromorphic function f, there is at least one bounded component of the f-preimage of one of these domains.

In [7] Ahlfors proved all these conjectures. His technically hard proof was a combination of topological considerations with sophisticated distortion theorems.

Finally he developed a general theory [8] which probably constitutes his most original contribution to the study of meromorphic functions. The theory has a metric-topological nature; all complex analysis in it is reduced to one simple application of the Length – Area Principle. This makes the theory flexible enough to treat quasiconformal mappings because for such mappings the Length – Area argument also works. In fact the word "quasiconformal" was for the first time used in this paper. In his commentary in [1] to [8] Ahlfors writes: "I included this more general situation in my paper but with pangs of conscience because I considered it rather cheap padding... . Little did I know at that time what an important role quasiconformal mappings would come to play in my own work".

We start with several definitions and elementary facts from the topology of surfaces. A bordered surface of finite type is a closed region on a compact orientable surface bounded by finitely many simple closed curves. (Compact orientable surfaces are spheres with finitely many handles attached). The Euler characteristic of a bordered surface χ is[8] $2g-2+k$, where g is the number of handles (genus) and k is the number of holes (boundary curves). We consider a topologically holomorphic map of two bordered surfaces of finite type, $\pi : R \to R_0$. If $\pi(\partial R) \subset \partial R_0$ then π is a ramified covering. In this case we have

(i) there is a number d, called the degree of π, such that every point in R_0 has d preimages, counting multiplicity;
(ii) $\chi(R) \geq d\chi(R_0)$.

In fact, a more precise relation holds: $\chi(R) = d\chi(R_0) + i(\pi)$, where $i(\pi)$ is the number of critical points of π, counting multiplicity. This is the Riemann–Hurwitz formula.

Ahlfors' theory extends these two facts to the case when π is *almost* a ramified covering, that is, the part of the boundary $\partial_0 R \subset \partial R$ which is mapped to the interior of R_0 is relatively small in the sense we are going to define now. This part $\partial_0 R$ is called the relative boundary.

Let R_0 be a region in a compact surface equipped with a Riemannian metric. A curve is called regular if it is piecewise smooth. A region is called regular if it is bounded by finitely many piecewise smooth curves.[9] We assume that R_0 is a regular region. The restriction of the Riemannian metric to R_0 is called ρ_0 and its pullback to R is called ρ.

Let L be the length of the relative boundary $\partial_0 R$. Everything on R is measured using the pulled back metric ρ. We use the symbol $|.|$ for the area of a regular region or for the length of a regular curve, depending on context. We define the *average number of sheets* of π by $S := |R|/|R_0|$. Similarly, if $X_0 \subset R_0$ is a regular region or curve, we define the average number of sheets over X_0 as $S(X_0) = |X|/|X_0|$, where $X = \pi^{-1}(X_0)$.

The following results are the First and Second Main Theorems of Ahlfors' theory.

(5.2) *For every regular region or curve $X_0 \subset R_0$, there exists a constant k depending only of R_0 and X_0 such that*

$$|S(X_0) - S| < kL.$$

[8]We follow Ahlfors' notations. In modern literature the Euler characteristic is usually defined with the opposite sign.

[9]Ahlfors defines the precise degree of regularity for which his theory works. Here we make stronger smoothness assumptions only for simplicity.

(5.3) *There exists a constant k depending only of R_0 such that*

$$\max\{\chi(R), 0\} \geq S\chi(R_0) - kL.$$

It is important in these theorems that k does not depend on R and π. They contain useful information if L is much smaller then S, when they imply that properties (i) and (ii) above hold approximately.

The proofs of (5.2) and especially (5.3) are rather sophisticated[10], though elementary. The proof of (5.3) was later simplified by Y. Toki [51].

Now we explain how this applies to open Riemann surfaces and meromorphic functions. Let us consider a meromorphic function

$$f : R \to \bar{\mathbf{C}},$$

where R is the plane $\mathbf{C}$ equipped with the pullback of the spherical metric ρ_0 on $\bar{\mathbf{C}}$. One can consider any compact Riemann surface with any Riemannian metric instead of $\bar{\mathbf{C}}$. A surface with a Riemannian metric R is called *regularly exhaustible* if there exists an increasing sequence of compact regular subregions $F_1 \subset F_2 \subset \ldots, \cup F_j = R$ with the property $|\partial F_j|/|F_j| \to 0$ as $j \to \infty$. It is easy to prove using the Length – Area Principle that $\mathbf{C}$ is always regularly exhaustible, no matter what the conformal metric is, and a sequence of concentric Euclidean disks can be always taken as F_j. Thus one can apply (5.2) and (5.3) to the restrictions of a meromorphic function f on the disks $|z| \leq r$. Then the average number of sheets $S = A_f(r)$ is the same $A(r)$ as in (2.4). We denote the length of the relative boundary by $L(r)$, this is nothing but spherical length of the f-image of the circle $|z| = r$. The Euler characteristic of the sphere is negative, $\chi(\bar{\mathbf{C}}) = -2$, so (5.3) gives nothing. To extract useful information from (5.3) we consider a region $R_0 \subset \bar{\mathbf{C}}$ obtained by removing from $\bar{\mathbf{C}}$ a collection of $q \geq 3$ disjoint spherical disks D_j, so that $\chi(R_0) = q - 2$. Now we apply (5.3) to the restriction of f on $R(r) := f^{-1}(R_0) \cap \{z : |z| \leq r\}$. The Euler characteristic of this region is the number of boundary curves minus two. A component of $f^{-1}(D_j) \cap \{z : |z| \leq r\}$ is called an *island* if it is relatively compact in $|z| < r$, and all other components are called *peninsulae.* If $\bar{n}(r, D_j)$ is the number of islands over D_j, then $\chi(R(r)) = \sum_j \bar{n}(r, D_j) - 1$. Thus from (5.2) and (5.3) we obtain

$$\sum_{j=1}^{q} \bar{n}(r, D_j) \geq (q-2)A(r) + o(A(r)) \tag{5.4}$$

for those r for which $L(r)/A(r) \to 0$. This result is called Scheibensatz. It has to be compared with the SMT of Nevanlinna in the form (4.3). First of all, (5.4) can be integrated with respect to r to obtain

$$\sum_{j=1}^{q} \bar{N}(r, D_j) \geq (q-2)T(r) + o(T(r)) \tag{5.5}$$

with a somewhat worse exceptional set for r than in (4.3) [42]. Now if a_j lie in the interiors of D_j for $j = 1, \ldots, q$ then evidently $\bar{N}(r, a_j) \geq \bar{N}(r, D_j)$ because the

[10] Ahlfors won one of the first two Fields Medals for this in 1936.

restriction of f on an island is a ramified covering over D_j. Thus (5.5) is better than (4.3) because (5.5) does not count a-points in peninsulae.

The Five Islands Theorem (5.1) follows from (5.4) in the same way as (4.4) follows from (4.3).

A substantial complement to this theory was made by Dufresnoy [24], who invented the way of deriving normality criteria from Ahlfors' theory using a simple argument based on an isoperimetric inequality (see also [34] for a very clear and self-contained exposition of the whole theory and its applications). The normality criteria we are talking about are expressions of another heuristic principle[11] sometimes associated with the name of Bloch (based on his paper [14])

(5.6) *that every condition which implies that a function meromorphic in* $\mathbf{C}$ *is constant, when applied to a family of functions in a disk, should imply that this family is normal, preferably with explicit estimates.*

For example:

Given five regular Jordan regions with disjoint closures, consider the class of meromorphic functions in a disk which have no simple islands over these regions. Then this class is a normal family.

In fact this result was obtained by Ahlfors himself in [6] but Dufresnoy's method permits an automatic derivation of many similar results from Ahlfors' metric-topological theory.

Another situation when (5.3) applies is the following. Let $D \subset \mathbf{C}$ be a domain, $f : D \to D_0$ a meromorphic function in $\bar{D}\backslash\{\infty\}$, and $D_0 \subset \bar{\mathbf{C}}$ a Jordan domain. Assume that f maps $\partial D \cap \mathbf{C}$ to ∂D_0, Then D equipped with the pullback of the spherical metric is regularly exhaustible; one can take an exhaustion by $D \cap \{z : |z| \leq r\}$. It follows from Ahlfors' theory that f can omit at most two values from D_0 [52], Theorem VI.9. This is how the result about countability of the set of direct singularities for a parabolic surface (3.3) can be proved. The situation when $D = D_0$ (and they are not necessarily Jordan) occurs frequently in holomorphic dynamics, namely in the iteration theory of meromorphic functions; see, for example [13], where D is a periodic component of the set of normality of a meromorphic function. In this case it follows from Ahlfors' theory that the Euler characteristic of D is non-positive [16].

Here is another variation on the same topic, due to Noshiro and Kunugui, see for example [52]:

let f *be a meromorphic function in the unit disk* $\mathbf{U}$ *such that* $\lim_{z\to\zeta} f(z)$ *exists and belongs to the unit circle* $\partial\mathbf{U}$ *for all* $\zeta \in \partial\mathbf{U}\backslash E$*, where* E *is a closed set of zero logarithmic capacity,* $|f(0)| < 1$*. Then* $f(z) = a$ *has solutions* $z \in \mathbf{U}$ *for all* $a \in \mathbf{U}$ *with at most two exceptions. If* f *is holomorphic then the number of exceptional values is at most one.*

In [9] Ahlfors made, using his own expression, his third attempt to penetrate the reasons behind Nevanlinna's value distribution theory. This time he bases his investigation on the Gauss – Bonnet formula, which is

$$\int\int_R K\, d\rho = -2\pi\chi - \int_{\partial R} g ds.$$

[11] Several rigorous results which may be considered as implementations of this principle are known: [61,40].

Here R is a bordered surface with a smooth Riemannian metric ds with associated area element $d\rho$, K is the Gaussian curvature, χ is the Euler characteristic and g is the geodesic curvature. Choosing a metric with finitely many singularities of the type (4.5) and pulling it back from R_0 to R via π, one can obtain the relation between Euler characteristics of R_0 and R. The only analytical problem is to estimate the integrals along the boundary curves in the Gauss–Bonnet formula. This general method permits one to give a unified proof of both (5.4) and the usual SMT (4.3). This paper [9] was the basis of many generalizations of value distribution theory.

6. Bloch constant and Ahlfors' extension of Schwarz' lemma.

In 1921 Valiron proved another necessary condition of parabolic type.
Let

$$\pi : R \to \mathbf{C}$$

be a parabolic Riemann surface. Let ds be the the pullback of the Euclidean metric from $\mathbf{C}$ to R. Then there are disks of arbitrarily large radii in R with respect to ds such that the restriction of π to these disks is one-to-one.
This can be easily derived from the Five Islands theorem (5.1). Much better known is the statement which corresponds to this via Bloch's Principle (5.6).

Bloch's Theorem. *Let f be a holomorphic function in the unit disk $\mathbf{U}$ satisfying $|f'(0)| = 1$. Then there is a relatively compact region $D \subset \mathbf{U}$ which is mapped by f univalently onto a disk of radius B, where B is an absolute constant.*

The largest value of B for which this theorem is true is called Bloch's constant. Its precise value is unknown to this day. A very plausible candidate for the extremal Riemann surface can be described as follows. Consider the tiling of the plane with equilateral triangles, such that 0 is the center of one of the triangles. Denote the set of all vertices by X. There is unique simply connected Riemann surface R which has no singular points over $\mathbf{C}$ and all points over X are simple critical points. This R is of hyperbolic type by Valiron's theorem. Let f be the corresponding function in the unit disk (normalized as in Bloch's theorem). In [10] the authors make the calculation for this function and find that

$$B \leq B' := \sqrt{\pi} \cdot 2^{1/4} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{4}\right)} \left(\frac{\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{12}\right)} \right)^{1/2} = 0.4719\ldots.$$

This B' is conjectured to be the correct value of Bloch's constant B.

In [11] Ahlfors gives the lower estimate $B \geq \sqrt{3}/4 > 0.433$, but much more important is the method which Ahlfors used to obtain this result.

Let $ds = p(z)|dz|$ be a conformal metric. Its Gaussian curvature is expressed by $-p^{-2}\Delta \log p$. This is invariant under conformal mappings. We denote by $d\sigma$ the Poincaré metric in the unit disk,

$$|d\sigma| = 2\frac{|dz|}{1-|z|^2},$$

whose Gaussian curvature is identically equal to -1. The usual conformal invariant form of the Schwarz lemma says that every holomorphic map of the unit disk into itself is contracting with respect to Poincaré metric. We call $ds' = p'|dz|$ a supporting metric of $ds = p|dz|$ at the point z_0 if $p'(z_0) = p(z_0)$ and $p'(z) \leq p(z)$ in a neighborhood of z_0.

Theorem. *Suppose p is continuous in the unit disk and it is possible to find a supporting metric of curvature ≤ -1 at every point. Then $ds \leq d\sigma$.*

The usual Schwarz lemma follows if we take the pullback of $d\sigma$ as ds. This extended form of the Schwarz lemma is frequently used now in differential geometry and geometric function theory. See [18] for a survey of these applications.

Returning to Bloch's constant, the best known lower estimate is $B > \sqrt{3}/4 + 2 \times 10^{-4}$, which is due to Chen and Gauthier [20].

7. Holomorphic curves.

From the algebraic point of view the Riemann sphere $\bar{\mathbf{C}}$ is the complex projective line $\mathbf{P}^1$. Value distribution theory can be extended to holomorphic mappings from $\mathbf{C}$ to complex projective space $\mathbf{P}^n$. Let us recall the definition. In $\mathbf{C}^{n+1}\backslash\{0\}$ we consider the following equivalence relation: $(z_0, \dots, z_n) \sim (z'_0, \dots, z'_n)$ if there is a constant $\lambda \in \mathbf{C}\backslash\{0\}$ such that $z'_j = \lambda z_j$. The set of equivalence classes with the natural complex analytic structure is n-dimensional complex projective space $\mathbf{P}^n$. Let $\pi : \mathbf{C}^{n+1} \to \mathbf{P}^n$ be the projection. Coordinates of a π-preimage of a point $w \in \mathbf{P}^n$ in $\mathbf{C}^{n+1}$ are called homogeneous coordinates of w. A holomorphic map $f : \mathbf{C} \to \mathbf{P}^n$ is called a holomorphic curve. It can be lifted to a map $F : \mathbf{C} \to \mathbf{C}^{n+1}$ such that $f = \pi \circ F$. This $F = (F_0, \dots, F_n)$ is called a homogeneous representation. Here the F_j are entire functions without common zeros. For $n = 1$ this is just a representation of a meromorphic function as a quotient of two entire ones. A generic point in $\mathbf{P}^n$ has no preimages under f so one studies preimages of hypersurfaces, in particular, preimages of hyperplanes. An analog of Nevanlinna's theory for this case was created by H. Cartan [19], and the role of the spherical metric is now played by a Hermitian positive $(1, 1)$-form, the so-called Fubini-Study form. The pullback of this form via a holomorphic curve f is

$$d\rho = \|F\|^{-4} \sum_{i>j} |F_i F'_j - F'_i F_j|^2 \, dm,$$

where $\|.\|$ is the usual Euclidean norm in $\mathbf{C}^{n+1}$ and dm is the area element in $\mathbf{C}$. The Nevanlinna-Cartan characteristic of f is defined by

$$T_f(r) = \int_0^t \frac{A_f(t)}{t} dt, \quad \text{where} \quad A(r) = \int_{|z| \leq r} d\rho.$$

A hyperplane $A \subset \mathbf{P}^n$ is given by a linear equation in homogeneous coordinates: $a_0 z_0 + \ldots + a_n z_n = 0$, so the preimages of this hyperplane under f are just zeros of the linear combination $g_A = a_0 F_0 + \ldots + a_n F_n$. The number of these zeros in the disk $|z| \leq r$ is denoted by $n(r, A)$ and the averaged counting functions $N(r, A)$ are defined as before. The proximity functions $m(r, A)$ can be also defined by analogy with the one-dimensional case, the role of the chordal distance now being played by the sine of the angle between a line and a hyperplane:

$$m(r, A) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{\|F\| \cdot \|A\|}{|g_A|} \, d\theta.$$

The First Main Theorem of Cartan says that

$$m(r, A) + N(r, A) = T(r) + O(1) \quad \text{for every hyperplane } A, \tag{7.1}$$

where we assume only that $f(\mathbf{C})$ is not contained in A. In the Second Main Theorem of Cartan we assume that several hyperplanes $A_1, \dots, A_q$ are in *general position*, that is, every $n+1$ of them have empty intersection. We also denote by $N^*(r)$ the averaged counting function of zeros of the Wronskian determinant $W(F_0, \dots, F_n)$; these zeros do not depend on the choice of homogeneous representation F. To guarantee that this Wronskian is not identical zero we need to assume that our curve is *linearly non-degenerate*, that is, $f(\mathbf{C})$ is not contained in any hyperplane, which is the same as to say that $F_0, \dots, F_n$ are linearly independent. The SMT of Cartan is

$$\sum_{j=1}^{q} m(r, A_j) + N^*(r) \leq (n+1)T(r) + S(r), \tag{7.2}$$

where the error term $S(r)$ has the same estimate as in one-dimensional Nevanlinna theory.

One important corollary is the Borel theorem: *a curve omitting* $n+2$ *hyperplanes in general position has to be linearly degenerate.* This is not a "genuine" Picard-type theorem, because the conclusion is "linearly degenerate" rather than "constant". But one can deduce from Borel's theorem the following: *a curve omitting* $2n+1$ *hyperplanes in general position has to be constant.* This corollary from Borel's theorem was apparently formulated for the first time by P. Montel [43].

It is strange that Cartan's paper [19] was overlooked by many researchers for about 40 years. Even as late as in 1975 some specialists in holomorphic curves were surprised to hear about Cartan's work.

Meanwhile H. Weyl and his son J. Weyl in 1938 restarted the subject of value distribution of holomorphic curves from the very beginning [57], independently of Cartan. They failed to come to conclusive results like Cartan's second main theorem, but they contributed two important ideas to the subject. The first of them was to consider the so-called *associated curves.* Here we need some definitions. A set of $k+1$ linearly independent vectors $A_0, \dots, A_k \in \mathbf{C}^{n+1}$ defines a $k+1$-dimensional linear subspace. It is convenient to use the wedge product $A_0 \wedge \ldots \wedge A_k$ to describe this subspace. Two $(k+1)$-tuples define the same subspace if and only if their wedge products are proportional. The elements of $\wedge^{k+1}\mathbf{C}^{n+1}$ are called polyvectors; those of them which have the form $A_0 \wedge \ldots \wedge A_k$, where $A_j \in \mathbf{C}^{n+1}$ are called decomposable polyvectors. If we fix a basis in $\mathbf{C}^{n+1}$ this defines a basis in $\wedge^{k+1}\mathbf{C}^{n+1}$. Thus to each $k+1$-subspace corresponds a point in $\mathbf{P}^{N_k}$, where $N_k = \binom{n+1}{k+1} - 1$, the homogeneous coordinates of this point being coordinates of $A_0 \wedge \ldots \wedge A_k$ in the fixed basis of $\wedge^{k+1}\mathbf{C}^{n+1}$. The coordinates we just introduced are called Plücker coordinates. A linear subspace of dimension $k+1$ in $\mathbf{C}^{n+1}$ projects into a projective subspace of dimension k. The set of all such projective subspaces is called Grassmanian $G(k,n)$. The Plücker coordinates identify $G(k,n)$ with a manifold in $\mathbf{P}^{N_k}$ of dimension $(k+1)(n-k)$. An explicit system of equations defining this manifold can be written (see for example [38,32]).

To every holomorphic curve f in $\mathbf{P}^n$, the Weyls associate curves $f_k : \mathbf{C} \to G(n,k)$ which in homogeneous coordinates have the representation $F \wedge F^{(1)} \wedge F^{(2)} \wedge \ldots \wedge F^{(k)}$, $k = 0, \dots, n-1$, where $F^{(j)}$ is the j-th derivative. The geometric interpretation is that they assign to each point $f(z)$ on the curve the tangent line $f_1(z)$ and the *osculating k-planes* $f_k(z)$. For each k, $f_k(z)$ is the unique k-dimensional projective subspace which has a contact of order at least k with the curve at the point $f(z)$.

The second important idea of the Weyls was to consider the proximity functions of a curve f with respect to projective subspaces of any codimension, not only with respect to hyperplanes. Such a proximity function may be obtained by averaging the hyperplane proximity functions over all hyperplanes containing the given subspace. The FMT (7.1) no longer holds for k-subspaces with $k < n-1$ because they normally have no preimages at all. But the SMT interpreted as an upper bound for proximity functions still has a sense. The Weyls [58] managed to prove such SMT for points, which are 0-subspaces.

It was Ahlfors who proved in [12] the precise estimates for proximity functions for all f_k in all codimensions. To formulate the result we use the following notation. Let A^h be a decomposable h-polyvector. Then

$$m_k(r, A^h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \frac{\|F_k\| \cdot \|A^h\|}{\|F_k \vee A^h\|} d\theta.$$

The expression in the logarithm is reciprocal to the distance between the polyvector F_k and the subspace orthogonal to A_h. It is expressed in terms of the inner product $\vee$ of polyvectors, which coincides with the usual dot product when $h = k$. The number of critical points of the map $f_k : \mathbf{C} \to G(n, k)$ is denoted by $n_k^*(r)$ and the corresponding averaged counting function by $N_k^*(r)$. We also set $T_k = T_{f_k}$. Now we assume that f is a linearly non-degenerate curve and a finite system of decomposable polyvectors in general position is given in each dimension h. Ahlfors' result is the following (we use the formulation from [60]).

For $0 \le h \le k$

$$\sum_{A^h} m_k(A^h) \le \binom{n+1}{h+1} T_k - \sum_{m=k}^{n-1}\sum_{i=0}^{h} p_h(m-i, h-i) N_{m-i}^* - \sum_{i=k-h-1}^{k} \binom{i}{h-k+i+1}\binom{n-i-1}{k-i} T_i + S(r),$$

and for $k \le h \le n-1$

$$\sum_{A^h} m_k(A^h) \le \binom{n+1}{h+1} T_k - \sum_{m=0}^{k}\sum_{i=0}^{n-k-1} p_{n-h-1}(n-m-i-1, m-h-i-1) N_{m+i}^* - \sum_{i=k}^{n-h+k} \binom{n-1-i}{h-k-1}\binom{i}{k} T_i + S(r),$$

where the following notation is used

$$p_h(k, l) = \binom{n+1}{h+1} - \sum_{j \ge 0} \binom{k+1}{k+j+1}\binom{n-k}{h-l-j} \ge 0.$$

Here the binomial coefficient is defined for all integers by $(1+x)^n = \sum_k \binom{n}{k} x^k$.

In particular for $h = k$, when the First Main Theorem applies we have

$$\sum_{A^k} m_k(A^k) \leq \binom{n+1}{k+1} T_k - \sum_{m=0}^{k} \sum_{i=m}^{n+m-k-1} p_k(i,m) N_i^* + S(r). \tag{7.3}$$

From here one can derive

$$\sum_{A^k} m_k(A^k) \leq \binom{n+1}{k+1} T_k. \tag{7.4}$$

This has the same form as Cartan's SMT (7.2) would give if one applies it to the associated curves f_k and drops the N^* term. But (7.4) *does not* follow from Cartan's SMT. The catch is that even if f is linearly non-degenerate, the associated curves f_k might be linearly degenerate. One can also show (see for example [28], p. 138), that the ramification term N^* in Cartan's SMT can be obtained from (7.3). Thus Ahlfors' result is stronger then Cartan's SMT when applied to the associated curves f_k, and decomposable hyperplanes, and they coincide when applied to the curve f itself and hyperplanes.

The difficulties Ahlfors had to overcome in this paper are enormous. The main idea can be traced back to his proofs of Nevanlinna's SMT, but multidimensionality causes really hard problems. As Cowen and Griffiths say in [21] "The Ahlfors theorem strikes us as one of the few instances where *higher codimension* has been dealt with *globally* in complex-analytic geometry". Ahlfors in his commentaries [1], p. 363, says: "In my own eyes the paper was one of my best, and I was disappointed that years went by without signs that it had caught on".

Since he wrote this the situation has changed a little. H. Wu [60] published a detailed and self-contained exposition of Ahlfors' work. Two new simplified versions of the proof of (7.3) were given in [21] and [28]; the second work gives important applications to minimal surfaces in $\mathbf{R}^n$. Still, from my point of view much in this work of Ahlfors remains unexplored. For example, let f be a holomorphic curve in $\mathbf{P}^2$ and $a_1, \ldots, a_q$ a collection of points in $\mathbf{P}^2$. What is the smallest constant K such that

$$\sum_{j=1}^{q} m_f(r, a_j) \leq K T_f(r) + S(r) \tag{7.5}$$

holds? Ahlfors' relations will give this with $K = 3/2$; the same can be deduced from (7.2). But one can conjecture that in fact $K = 1$. A related question was asked by Shiffman in [48]; no progress has been made.

8. Multi-dimensional counterpart of the type problem.

Here we will mention very briefly some later research in the spirit of the type problem where Ahlfors' ideas play an important role.

Quasiregular mappings of Riemannian manifolds. A mapping f between two n-dimensional Riemannian manifolds is called K-quasiregular if it belongs to the Sobolev class $W^1_{n,\mathrm{loc}}$ and its derivative almost everywhere satisfies $|df|^n \leq K J_f$, where J_f is the Jacobian. Given two orientable Riemannian manifolds V_1 and V_2, one may ask whether a non-constant quasiregular map $f : V_1 \to V_2$ exists. To

answer this question, Gromov [33, Ch. 6] generalizes the parabolic type criterion from Section 1 in the following way. Let $d(V)$ be the supremum of real numbers $m > 0$ such that for some constant C the isoperimetric inequality volume$(D) \leq C$ area$(\partial D)^{m/(m-1)}$ holds for every compact $D \subset V$. Here "area" stands for the $n-1$ dimensional measure. This number $d(V)$ is called *isoperimetric dimension* of V. The isoperimetric dimension of $\mathbf{R}^n$ with the Euclidean metric is n and the isoperimetric dimension of the hyperbolic (Lobachevskii) space $\mathbf{H}^n$ is ∞. Ahlfors' argument presented in Section 1 proves the following:
Assume that for some point $a \in V$ we have

$$\int^{\infty} \frac{dr}{L^{1/(m-1)}(r)} = \infty, \tag{8.1}$$

where $L(r)$ is the $(n-1)$-measure of the sphere of radius r centered at a. Then the isoperimetric dimension of any conformal metric on V is at most m.

As a corollary, one deduces that a nonconstant quasiregular map from V_1 to V_2, where V_1 satisfies (8.1), is possible only if the isoperimetric dimension of V_2 is at most m.

Because f can be lifted to a map $\tilde{f} : V_1 \to \tilde{V}_2$ to the universal covering, one is interested in estimating $d(\tilde{V}_2)$. Gromov proves that for compact V_2 the isoperimetric dimension of $\tilde{V}_2$ depends only on the fundamental group: a distance and volume can be introduced on every finitely generated group, and the fundamental group will satisfy the same isoperimetric inequality as $\tilde{V}_2$.

As a simple corollary one obtains

(8.2) *There is no quasiregular map from $\mathbf{R}^3$ to $S^3 \backslash N$, where S^3 is the 3-dimensional sphere and N is a non-trivial knot.*

This result is superseded by the following generalization of Picard's theorem proved by Rickman in 1980

(8.3) *For every $n \geq 2$ and $K \geq 1$ there exists q such that a K-quasiregular map $\mathbf{R}^n \to S^3$ cannot omit more then q points.*

Holopainen and Rickman [39] later proved that S^3 can be replaced with any compact manifold. The simplest proof of (8.3) is due to Lewis [41]. It is also one of the most elementary proofs of Picard's theorem.

The experts were stunned when Rickman constructed examples [47] showing that the number of omitted values may really depend on K. Any finite number of values can be actually omitted.

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[12]In items [2-11] the last number corresponds to numeration in [1].

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