Here is how this can be done for a ring. Imagine a two-sheeted Riemann surface over the unit disk, with branch points at $t, -t$, where $0 < t < 1$. This Riemann surface is conformally equivalent to a ring (by the Riemann Hurwitz) and the modulus of this ring is a function of t. When $t \to 0$, the modulus tends to 0, when $t \to 1$, it tends to infinity. So for various t we obtain all rings, and this construction gives the Ahlfors function of the ring.

Now modify our surface. On each of the two sheets make a vertical cut from 0 to i. And glue a $d_1 - 1$ sheeted disk with the similar cut along the cut to the first sheet and $d_2 - 1$ sheeted disk to the second sheet. You obtain a Riemann surface whose boundary consists of a d_1 sheeted circle and a d_2-sheeted circle, both lying over the unit circle. It has 4 branch points: two over 0 connecting d_1 and d_2 sheets, respectively, and two over $t, -t$ connecting two sheets each.

It is clear that this new surface is a ring. (By Riemann-Hurwitz, or just by examining the gluing procedure).

Moving $t > 0$ from 0 to 1, we obtain rings of all possible moduli, by an easy argument with extremal length.

This proves your conjecture for a ring.