

# An Asymptotic Slope Conjecture for Finite-Type Entire Functions

Working formulation, September 2026

## Background

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a transcendental entire function with finitely many singular values, and assume that  $f$  has finite positive order  $\rho(f)$ . Such functions form the Speiser class  $\mathcal{S}$ . Their quasiconformal deformation space carries a natural Teichmüller metric.

If  $f$  and  $g$  are  $K$ -quasiconformally equivalent finite-order entire functions, the standard distortion estimate gives

$$\frac{1}{K} \leq \frac{\rho(g)}{\rho(f)} \leq K.$$

Thus, if  $t \mapsto f_t$  is a unit-speed Teichmüller geodesic and the convention

$$d_T(f_s, f_t) = |s - t| = \frac{1}{2} \log K$$

is used, then

$$|\log \rho(f_t) - \log \rho(f_s)| \leq 2|t - s|.$$

Hence  $\log \rho$  is always 2-Lipschitz along such a geodesic.

There are two contrasting phenomena. Under the area property of Epstein–Rempe–Gillen, the order is invariant under topological (hence quasiconformal) deformation, so the asymptotic slope is 0. On the other hand, Bishop constructed a finite-order entire function with only three singular values whose order changes under quasiconformal equivalence. Reading his construction with the dilatation parameter left free suggests deformations for which

$$\log \rho(f_t) = 2t + O(1),$$

so that the maximal slope allowed by the general Lipschitz bound is asymptotically attained.

These two extremal behaviors motivate the following question: although  $\log \rho(f_t)$  need not be expected to be convex, must it nevertheless have a well-defined linear growth rate along every Teichmüller ray?

**Asymptotic Slope Conjecture.** *Let  $f$  be a finite-order transcendental entire function with finitely many singular values, and let*

$$t \longmapsto f_t, \quad t \geq 0,$$

*be a unit-speed Teichmüller geodesic ray in the quasiconformal deformation space of  $f$ . Then the limit*

$$\boxed{\lim_{t \rightarrow \infty} \frac{\log \rho(f_t)}{t}}$$

*exists.*

Equivalently, there should exist a number  $\alpha = \alpha(f)$  such that

$$\log \rho(f_t) = \alpha t + o(t) \quad (t \rightarrow \infty).$$

The general quasiconformal distortion bound forces  $\alpha \leq 2$ . The known invariant-order situation gives  $\alpha = 0$ , while Bishop-type deformations suggest that  $\alpha = 2$  can occur.

## Why this may be natural

The order of an entire function can be read from the asymptotic growth of its covering surface, while a Teichmüller ray is governed by the asymptotic distortion of extremal lengths in a fixed measured-foliation direction. This suggests that Ahlfors' theory of covering surfaces could provide the missing bridge: one would seek to express the radial growth of the lifted covering in terms of extremal-length data and then use the asymptotic geometry of Teichmüller rays to extract a linear slope.

A stronger possible form would assert

$$\log \rho(f_t) = \alpha t + O(1),$$

but the existence of the limit above is the more conservative first conjecture.

## References

1. C. J. Bishop, *The order conjecture fails in  $\mathcal{S}$* , 2020.
2. A. L. Epstein and L. Rempe-Gillen, *On invariance of order and the area property for finite-type entire functions*, Ann. Acad. Sci. Fenn. Math. **40** (2015), 573–599.