Why airplanes fly, and ships sail

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April 23, 2013

And windmills rotate and propellers pull, etc... .

Denote $z = x + iy$ and let $v(z) = v_1(z) + iv_2(z)$ be the velocity field of a stationary flow in a plane region.

We consider a non-rotational flow of an incompressible fluid which means

$$\operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \tag{1}$$

and

$$\operatorname{curl} v = \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} = 0. \tag{2}$$

Introducing the *complex velocity* or *velocity function*

$$f(z) = \overline{v(z)} = v_1(z) - iv_2(z), \tag{3}$$

we notice that (1) and (2) are exactly the Cauchy–Riemann conditions for f. Thus the conjugate velocity is analytic if and only if the flow is irrotational and incompressible.

Euler's equation is the continuous version of the Newton's equation: it says that acceleration of fluid particles is equal to the force. First we compute the acceleration. Let $Q(t) = (x(t), y(t))$ be the trajectory of a particle, then the velocity is $dQ/dt = v(Q(t))$, and

$$\begin{aligned} \frac{d^2Q}{dt^2} &= \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial v}{\partial x}v_1 + \frac{\partial v}{\partial y}v_2 = (v \cdot \nabla)v. \end{aligned}$$

Acceleration times mass equals force. The force comes from the pressure $p(z)$, a scalar (real valued) function. The mass is described by the density ρ. So we obtain the Euler equation

$$\rho(v \cdot \nabla)v = -\nabla p.$$

Minus because the force acts from the place of larger pressure towards the place with smaller pressure. For constant ρ and irrotational flow this has a first integral

$$\frac{\rho}{2}|v|^2 + p = c, \tag{4}$$

which is called *Bernoulli's law*. (Daniel Bernoulli). Indeed, when we compute $\nabla|v|^2$ using (2), we obtain $2(v \cdot \nabla)v$.

We are interested in the flow past an obstacle. Our obstacle is a rigid body whose boundary is a piecewise-smooth curve γ. We assume that γ is parametrized counter-clockwise. Then the flow is described by an analytic function f in the exterior region of γ, and we assume that $f(\infty) = v_\infty$, velocity at infinity, is a positive number. This means that the flow is directed horizontally left to right.

The *lifting force* L is the total force with which the flow acts on the body. It comes from the pressure of the flow.

At every point of γ this pressure is perpendicular to γ, and directed inwards. The pressure on a small piece of γ is thus given by the formula $(i\rho/2)(c - |v^2|)dz$ which follows from Bernoulli's law. But integral of a constant over a closed curve is zero, so we only need to integrate

$$-(i\rho/2)|v|^2 dz.$$

Writing $dz = |dz|e^{i\phi}$ on γ, we obtain $v = \pm|v|e^{i\phi}$ Indeed, velocity on γ must be tangent to γa because the wing is impenetrable for the flow. So $v^2 = |v^2|e^{2i\phi}$, and $v^2 d\overline{z} = |v|^2 dz$. So the lifting force equals

$$L = -\frac{i\rho}{2}\int_\gamma v^2(z)d\overline{z} = -\frac{i\rho}{2}\int_\gamma \overline{f(z)}^2 d\overline{z},$$

or

$$\overline{L} = \frac{i\rho}{2}\int_\gamma f^2(z)dz. \tag{5}$$

This is called the *Blasius–Chaplygin formula*.

We can compute the integral using the residues. If

$$f(z) = c_0 + c_{-1}/z + \dots,$$

then

$$f^2(z) = c_0^2 + 2c_0 c_{-1}/z + \dots,$$

and

$$\int_\gamma f^2(z)dz = 2\pi i \cdot 2c_0 c_{-1} = 4\pi i c_0 c_{-1}.$$

Let us determine the physical meaning of the coefficients. $c_0 = f(\infty) = v_\infty$ is the velocity of the flow at infinity. Now $-c_{-1}$ is the residue:

$$c_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_\gamma v_1 dx + v_2 dy + \frac{1}{2\pi}\int_\gamma v_1 dy - v_2 dx.$$

The first integral is the circulation of the flow around γ, we denote it by C, while the second integral is the flux through γ. As the curve γ is impenetrable for the fluid, the flux is zero. So the final result is

$$L = -i\rho v_\infty C.$$

This is called the *Kutta–Joukowski* theorem. In words: *The lifting force is the product of the density, velocity and circulation, and directed perpendicular to the flow.* If the flow is left to right, *negative*, (clockwise) circulation gives the lifting force directed up.

This explains in principle why airplanes fly, why propellers pull or push, and why sailing boats can sail at various angles to the wind, etc.

The problem remains, how to find the circulation C for the given profile. We begin with the more general problem of determining the complex velocity for a given obstacle.

In any simply connected region occupied by the flow, we can consider the primitive F of the velocity function f. This function is defined up to an additive constant. It is called the *complex potential.* To see its physical meaning, we write

$$F = F_1 + iF_2, \quad f = F' = \partial F_1/\partial x - i\partial F_1/\partial y = v_1 - iv_2, \tag{6}$$

so $v = \nabla F_1$, and $f = F'$. This implies that velocity is perpendicular to the level lines of F_1. This justifies the name *velocity potential* for F_1. Thus the velocity is tangent to the level lines of $F_2 = \Im F$. This can be also seen by differentiating F_2 along the flow:

$$\frac{dF_2}{dt} = \frac{\partial F_2}{\partial x} v_1 + \frac{\partial F_2}{\partial y} v_2 = \frac{\partial F_2}{\partial x}\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\frac{\partial F_1}{\partial y} = 0,$$

where we used (6) and the Cauchy-Riemann conditions for F. The imaginary part F_2 of the complex potential is called the *stream function* and its level lines are called the *stream lines*.

In the case that our profile is connected and bounded (monoplane wing[1]), the region occupied by the flow is doubly connected, and the complex potential is in general multiple-valued. In the case of a flow around a smooth obstacle the boundary of the obstacle must be a stream line. The expansion of F at infinity

$$F(z) = v_\infty z + c_{-1} \log z + \dots$$

is obtained by integration of the expansion of f. Such a function F is determined up to an additive constant by γ and by the numbers v_∞ and c_{-1}. Indeed the difference of two such functions would be analytic in the outside region of γ, zero at infinity and its imaginary part will be constant on γ.

When γ is a single curve, one such F with $c_{-1} = 0$ exists by the Riemann mapping theorem. Indeed, take for F the conformal map of the exterior of γ (including infinity) onto the exterior region of a horizontal segment, normalized so that $F(z) \sim z,\ z \to \infty$. (This is called *the hydrodynamic normalization* btw, it is equivalent to the normalization in the Riemann mapping theorem; the role of 0 is now played by ∞).

So we always have a complex potential F_0 with zero circulation.

Example 1. A zero-circulation flow past a cylinder. We take $F(z) = z + 1/z = 2J(z)$, where J is the Joukowski function. The complex velocity is $f(z) = 1 - 1/z^2$. In this example, $v_\infty = 1,\ C = 0,\ L = 0$. The flow is symmetric with respect to the real line.

Example 2. Non-zero circulation flow past a cylinder.

$$F(z) = z + 1/z + ai \log z$$

This is multiple-valued, but notice that the complex velocity $f = F'$ is well defined, and the stream lines are also well defined, because the branches of F

[1]The case of biplane wing is even more interesting mathematically.

differ by *real* additive constants. Also all branches of F send the unit circle to horizontal segments.

For this function, the circulation is $-2\pi a$, so positive a corresponds to a lifting force directed upwards. To sketch the flow we find the critical points $f(z) = 0$. We have

$$f(z) = 1 - z^{-2} + ia/z = 0, \quad z^2 + iaz - 1 = 0, \quad z_{1,2} = -ia/2 \pm \sqrt{1 - a^2/4}. \tag{7}$$

When $0 \leq a < 2$ both critical points lie on (the lower half of) the unit circle. The flow is qualitatively the same as in Example 1, but the symmetry is broken. If $a = 2$ the critical points coincide, and if $a > 2$ they are pure imaginary, but only one of them is in the region occupied by the flow (that is outside of the unit circle). This permits to sketch the flow.

Now we can find all possible flows with given circulation around any given (connected) obstacle by finding a conformal map from the flow region onto the exterior of the unit disc and transplanting the flow of Example 2. If ϕ is the conformal map and f the velocity field from Example 2, then

$$(f \circ \phi)\phi' \tag{8}$$

is the transplanted velocity field. (The usual rule of mapping vectors).

All this leaves open the question how to find the circulation for the given γ.

For this we have the *Joukowski-Kutta hypothesis*. (It is called the Chaplygin condition in [2, 3, 4]). A realistic airfoil profile has a sharp edge in the back. Sharp edge means an interior angle $\pi\alpha$ in the flow region, with $\alpha \in (1, 2]$. The conformal map ϕ of the flow region onto the exterior of the unit disc is thus of the form $\phi(z) \sim (z - c)^{1/\alpha}$ where c is the coordinate of the edge. So the derivative $\phi'(c)$ is infinite at this point. The only way to compensate this is to have $f(\phi(c)) = 0$, and Joukowski's hypothesis is that this always happens in the real flows. This hypothesis permits to compute the circulation and the lifting force for any such given profile. Indeed, the curve γ uniquely defines the normalized conformal map ϕ, thus it defines the point $\phi(c)$, and this $\phi(c)$ should coincide with the point z_2 of Example 2, according to Joukowski's hypothesis. So we can compute parameter a of Example 2 and thus the circulation, the lifting force, and the whole velocity field. One only needs to know the conformal map ϕ. This map can be always approximated numerically for a given curve γ. The hypothesis is well verified experimentally: under certain conditions, the flow indeed leaves the airfoil at the sharp edge.

Example 3. Suppose that our profile is the segment $[e^{-i\alpha}, -e^{-i\alpha}]$, and as always $v_\infty = 1$. The "sharp edge at the back" is $e^{-i\alpha}$. This angle α is called the *angle of attack*.

We map conformally the complement of the profile onto the complement of a disc, so that the map is normalized at infinity. The map is unique and is given by the formula

$$z \mapsto e^{-i\alpha} J^{-1}(ze^{i\alpha}),$$

where J is the Joukowski function. The right sharp edge is mapped to the point $e^{-i\alpha}$. We compose this map with the map of the Example 2, and the condition that the stream lines leave the profile at the edge gives

$$a = -2 \sin \alpha.$$

Thus, *for small angles of attack, the lifting force is proportional to the angle of attack*.

It is somewhat embarrassing that this theory, which belongs to Kutta and Joukowski was published only in 1902.

The only mathematical theory that existed before that was due to Newton. Newton considered the following model. The air consists of hard balls which elastically collide with the lower face of the inclined plane, as in Example 3, and this creates a pressure on this lower face, creating the lifting force. This theory gives that the lifting force is proportional to the *square* of the angle of attack.

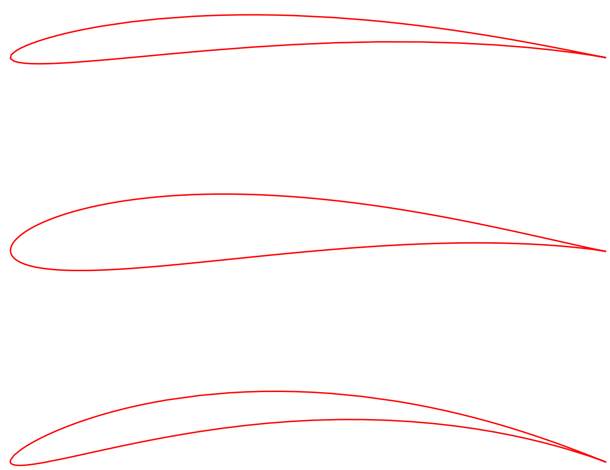
Some historians of science speculate [1] that Newton's theory actually delayed the development of aviation. However by the time when Lilienthal and the Wright brothers started their experiments with gliders, is was well-known that Newton's theory cannot be correct.[2]

Joukowski found a 2-parametric family of realistic profiles for which the conformal map ϕ is an elementary function (Joukowski's function, familiar to every student of Complex Variables). They are obtained in the following way. Consider the point $-a + ib$ where a and b are positive numbers and $a^2 + b^2 < 1$. Let K be the circle of radius $\sqrt{(1+a)^2 + b^2}$ centered at the

[2]There is an interesting recent twist in this story. It was found that Newton's theory better describes the flight at *hypersonic* speeds.

point $-a + ib$. This circle passes through 1. Joukowski map maps K onto some curve which has a sharp edge at the image of the point 1. The exterior angle at this point (which is interior for the region occupied by the flow) equals 2π because Joukowski's function has a critical point at 1 and doubles all angles at this point.

From the description above, it is easy to write a short program in Maple or any other language which will draw Joukowski's profiles. They are simple algebraic curves (images of circles under $z + 1/z$), and the old aerodynamic books contain recipes for their construction with compass and ruler, see, for example a beautiful picture in [6]. Joukowski profiles were actually used for several airplane wings http://www.ae.illinois.edu/m-selig/ads/aircraft.html.



Some Joukowski profiles.

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