Airplane and the wind. Solution

Using the hints, we introduce the coordinate system and denote by $y = f(x)$ the function whose graph is the trajectory of the airplane. When the airplane is at the point (x, y) on this graph, its velocity with respect to land is the sum of two vectors: wind velocity $(0, 1)$, and the airplane's velocity with respect to air. The last vector has the same direction as $(1 - x, -y)$, and its length is k. Thus (after a simple calculation) the slope of the velocity vector with respect to land is

$$\frac{-ky + \sqrt{(1-x)^2 + y^2}}{k(1-x)}.$$

This has to be equal to $y' = dy/dx$, so we obtain the differential equation

$$ky'(1-x) = -ky + \sqrt{(1-x)^2 + y^2}.$$

A natural change of the independent variable $u = 1 - x$, $dy/dx = -dy/du$ transforms it to a homogeneous first-order equation:

$$y' = \frac{y}{u} - \frac{1}{k}\sqrt{1 + (y/u)^2}.$$

Putting $w = y/u$, $y' = w + uw'$, we obtain

$$uw' = -\frac{1}{k}\sqrt{1 + w^2},$$

which is separable,

$$\int \frac{dw}{\sqrt{1 + w^2}} = -\frac{du}{ku},$$

using the hyperbolic sine notation,

$$\sinh^{-1} w = -\frac{1}{k} \log|u| + C.$$

Using the initial condition $w = 0$ for $u = 1 - x = 1$, we conclude that $C = 0$.

So

$$w = \sinh\left(-\frac{1}{k}\log|u|\right) = \frac{1}{2}\left(|u|^{-1/k} - |u|^{1/k}\right),$$

and

$$y = \frac{u}{2}\left(u^{-1/k} - u^{1/k}\right).$$

When $k = 1$, that is the speed of the airplane with respect to air is equal to the speed of the wind, we obtain

$$y = \frac{1}{2}(1 - u^2) = x - \frac{x^2}{2}.$$

this equation defines a parabola with intercepts at $(0, 0)$ and $(2, 0)$, but only the part of this parabola from $(0, 0)$ to $(0, 1/2)$ has physical meaning. At the point $(0, 1/2)$ the airplane will be *at rest* with respect to land! In other words, $(0, 1/2)$ is an equilibrium. It follows that the airplane will never actually reach this point, slowing down when it approaches it. And of course it will never reach the destination, missing it by approximately 50 miles to the North.

On the other hand, when the speed of the airplane $k > 1$, it will reach the destination, because we have $y(1) = 0$ in this case. It is easy to understand that the time required to reach the destination increases indefinitely as $k \to 1+$. Finally, when $k < 1$, we have $y(x) \to +\infty$ as $x \to 1-$, which means that the airplane will be blown away by the wind in the direction of y-axis.

Remarks. Notice that the expression for w as a function of u coincides with the expression for v as a function of x in the Little Jo problem. In fact, if we look at this airplane from a baloon, released at at time 0 at the place of destination, then the airplane's motion becomes the same as Little Jo's motion, and the destination's motion same as the pig's motion!