Math 452/525 final exam, fall 2020

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You may use the textbook from this course, or your notes. *But no help from a human, internet, or computer/calculator.*

For each problem, please enter your ANSWER in the indicated place, and then show your work.

1. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}.$$

(The answer should be a real number).

ANSWER: $(\pi/\sinh\pi - 1)/2$.

2. Find the radii of convergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{5^n}{n} z^{2n}$,

ANSWER: $1/sqrt5$.

b) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$,

ANSWER: $1/e$.

c) $\sum_{n=1}^{\infty} \frac{n!}{10^n} z^n$,

ANSWER: 0.

d) $\sum_{n=0}^{\infty} a_n z^n = \frac{e^z}{z^2 + z + 2}$,

ANSWER: $\sqrt{2}$.

e) $\sum_{n=0}^{\infty} a_n z^n = \frac{\operatorname{Log}(z+2)}{z+1}$.

ANSWER: 2. (Zero of the denominator at $z = -1$ cancels since $\operatorname{Log} 1 = 0$).

3. Evaluate the integral

$$\int_0^\infty \frac{\cos(x\sqrt{2})}{x^4+1}\,dx.$$

(The answer should be a real number).

ANSWER: $\frac{\pi}{2e\sqrt{2}}(\sin 1 + \cos 1) = \frac{\pi}{2e} \sim (1 + \pi/4)$.

4. Evaluate the integral

$$\int_{|z|=2} (z^3 + z) \cos\left(\frac{1}{z}\right) dz,$$

where the circle is described counterclockwise.

ANSWER: $-11\pi i/12$.

5. For which complex values of a the equation

$$\cot z = a$$

has no complex solutions?

ANSWER: $a = 1$ and $a = -i$.

6. a) Find the conformal map f of the region

$$\{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0,\ |z| < 1\}$$

onto the upper half-plane, such that

$$f(i) = 0, \quad f(0) = 1, \quad f(1) = \infty.$$

b) Find $f((1+i)/2)$.

ANSWER: a) $f(z) = \frac{J(z^2)+1}{J(z^2)-1}$, but there can be other forms of the answer.
b) $(-7+24i)/25$.

7. How many solutions does the equation

$$e^{z-2} = z^2$$

have in the unit disk $\{z : |z| < 1\}$?

ANSWER: 2.

8. Does there exist a real function $u(x, y)$ such that the function

$$f(x + iy) = u(x, y) + i(x - x^3 + 3xy^2)$$

is analytic in the whole plane?

If the answer is "no", explain why; if the answer is "yes", find this function u.

ANSWER: Yes. $u(x, y) = -y - y^3 - 3x^2y + c$.