Sleeping armadillos and Dirichlet's Principle

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Glasser and Davison [1, 2] discussed the following problem.

Consider two disjoint balls B_1 and B_2 of unit radius in 3-space, and let $r > 1$ be the distance between their centers. Let u be a harmonic function in the complement of the balls, which takes value 1 on the surfaces of the balls, and $u(x) \to 0$ as $x \to \infty$. Denote

$$C(r) = \int_{\partial B_1 \cup \partial B_2} \frac{\partial u}{\partial n},$$

where $\partial/\partial n$ is the differentiation along the inward normal.

The problem stated in [1, 2] is *to prove that* $C(r)$ *is an increasing function on* $[1, \infty)$.

An interpretation of $C(r)$ given in [2] is the rate of heat loss by a pair of sleeping animals. Each animal is represented by a ball whose surface temperature equals 1, while the outside temperature is zero (far away from the bodies). Then $C(r)$ represents the rate of heat loss, and monotonicity of $C(r)$ explains the habit of many animals (like hedgehogs or armadillos) to huddle together at night to keep themselves warm.

In [2], an explicit expression was derived,

$$C(r) = \text{const} \sum_{k=1}^{\infty} \frac{(-1)^k}{U_k(r)}, \qquad (1)$$

where U_k are Chebyshev polynomials of the second kind, and monotonicity of $C(r)$ was verified numerically. The problem of proving monotonicity of the expression (1), stated in [1, 2] looks hard. Steven Finch, in a personal message, informed me that someone even tried to prove monotonicity of (1) using computer algebra!

Explicit expression (1) does not help much in proving the monotonicity. However, if one uses the original definition of $C(r)$, monotonicity follows easily from the basic principles of Classical Potential Theory, see, for example, [3], and in fact, a much more general result can be obtained.

I think this is a good example showing the power of general principles as compared to explicit computations.

Let $E = E(r)$ be the union of the two spheres. The constant $C(r)/4\pi$ is nothing but the (Newtonian) capacity $\operatorname{cap} E(r)$. It can be obtained by solving the following extremal problem for measures μ on E:

$$\frac{4\pi}{C} = \inf\{I(\mu):\ \mathrm{supp}\mu \subset E,\ \mu(E) = 1\}, \tag{2}$$

where

$$I(\mu) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|} \tag{3}$$

is the *energy* of a measure μ. A unique extremal measure exists for every compact set E of positive capacity. This extremal measure is called the equilibrium measure of E. In fact, u is the potential of this equilibrium measure, divided by capacity; this is a form of the Dirichlet's Principle, see, for example [3, Theorem 7.1c].

Monotonicity of $C(r)$ is a consequence of the following proposition which is is well-known and easy to prove:

Proposition *Let $\phi : E_1 \to E_2$ be a one-to-one map of compact sets in $\mathbf{R}^3$, which decreases distances, that is $|\phi(x) - \phi(y)| \leq |x - y|$. Then* $\operatorname{cap} E_2 \leq \operatorname{cap} E_1$.

Indeed, if we replace a measure by its image under a map that decreases distances, the energy (3) will evidently increase, so the solution of extremal problem (2) for E_2 is greater then or equal to that for E_1, and thus $\operatorname{cap} E_2 \leq \operatorname{cap} E_1$.

Here is a more formal proof. Let $M(E)$ denote the set of Borel measures satisfying $\operatorname{supp} \mu \subset E$ and $\mu(E) = 1$. Denote by

$$\phi_* : M(E_1) \to M(E_2), \tag{4}$$

which is defined by

$$\phi_*\mu(X) = \mu(\phi^{-1}(X),$$

for every Borel subset $X \in E_2$. It is clear that the map (4) is one-to-one. Now we have

$$\begin{aligned} I(\phi_*\mu) &= \int\int \frac{d\phi_*\mu(x)d\phi_*(y)}{|x-y|} = \int\int \frac{d\mu(x)d\mu(y)}{|\phi(x)-\phi(y)|} \\ &\geq \int\int \frac{d\mu(x)d\mu(y)}{|x-y|} = I(\mu). \end{aligned}$$

Thus

$$\inf_{\mu \in M(E_2)} I(\mu) \geq \inf_{\mu \in M(E_1)} I(\mu),$$

so $\operatorname{cap} E_2 \leq \operatorname{cap} E_1$. □

For our original problem this means that $C(r)$ is increasing (take ϕ which is identity on S_1 and moves S_2, as a rigid body, towards S_1). It is easy to show that actually $C(r)$ is strictly increasing.

Neither the number of animals nor their shape are relevant for this argument: the total rate of the energy loss by a group decreases when the animals come closer. The exact meaning of "coming closer" is described in the Proposition above. However the assumption that all animals have equal body temperature is important.

The above proposition establishes monotonicity of the *total* rate of loss of heat by the whole group of animals. In the original problem with two equal balls, the rate of loss of heat by one ball is of course 1/2 of this total.

In more general cases, it seems more interesting from the point of view of animal behavior, and more challenging mathematically, to find under what conditions one can assert that as the animals come closer together, the rate of heat loss decreases *for each individual animal*.

Indeed, it is its own rate of loss of heat that an individual animal feels, and the behavior we discuss is probably driven by individual feelings rather then some abstract "community goal".

I suppose that in general the individual rates of loss of heat might not be monotone in the sense of the above Proposition, but in simple situations this could be the case. The following two cases seem to be the simplest:

a) two balls of unequal radii, and

b) three balls of equal radii.

Is it true that if we move such balls closer (such that all pairwise distances between their centers decrease) the rate of heat loss for each ball will decrease?

References

[1] M. L. Glasser, Problem 77-5, SIAM Review19 (1977), 148.

[2] M.L. Glasser and S. G. Davison, A bundling problem, SIAM review 20 (1978), 178–180.

[3] J. Wermer, Potential Theory, Springer, NY 1974. (Lect. Notes Math., 408.)

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