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ON ASYMPTOTIC CURVES OF ENTIRE FUNCTIONS OF FINITE ORDER

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ABSTRACT. For any ρ, $0 < \rho \leqslant \infty$, there exists an entire function of order ρ such that for any asymptotic curve Γ on which $f \to \infty$ the relation $l(r, \Gamma) = O(r)$, $r \to \infty$, does not hold, where $l(r, \Gamma)$ is the length of that part of Γ contained in the disc $\{z: |z| < r\}$. The same is true of asymptotic curves on which $f \to a \neq \infty$ under the natural restriction that $1/2 < \rho < \infty$. This disproves a well-known conjecture of Hayman and Erdős. Several closely related results are obtained.

Bibliography: 24 titles.

In a lecture given at Moscow University in 1960, W. K. Hayman [1] stated the following conjecture. Let f be an entire function of finite order ρ. Then it is always possible to choose an asymptotic curve Γ on which f tends toward ∞ such that

$$l(r, \Gamma) = O(r), \qquad r \to \infty, \tag{0.1}$$

where $l(r, \Gamma)$ is the length of that part of Γ which is contained in the disc $\{z: |z| \leqslant r\}$. And for $\rho = 0$, Hayman conjectured the stronger condition

$$l(r, \Gamma) = r + o(r), \qquad r \to \infty. \tag{0.2}$$

Thirteen years later, Paul Erdős (see [2], Problem 2.41) repeated the same problem almost word-for-word, also posing as an open question the problem of whether (0.1) is also satisfied for those asymptotic curves on which the entire function f tends toward a finite asymptotic value.

In [3], Hayman has shown that if, for an entire function f,

$$\ln M(r, f) = O(\ln^2 r), \qquad r \to \infty, \tag{0.3}$$

then it is possible to take a ray $\{z: \arg z = \theta\}$ for almost all $\theta \in [0, 2\pi]$ as the asymptotic curve on which f tends toward ∞ (here and in what follows we use without explanation the standard notation of the theory of entire and meromorphic functions; see, for example, [4]). Thus if (0.3) is satisfied it is always possible to choose the asymptotic curve not only so that (0.2) holds, but even so that $l(r, \Gamma) \equiv r$.

In §1 of this paper it is shown that, for any function $\varphi(r)$ which tends toward $+\infty$ as $r \to +\infty$, it is possible to find an entire function f such that

$$\ln M(r, f) = O(\varphi(r) \ln^2 r), \qquad r \to \infty, \tag{0.4}$$

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and, for any asymptotic curve Γ on which f approaches ∞, (0.1) is not satisfied. Thus on the one hand, Hayman's conjecture is disproved, and on the other, it is shown that it is impossible to weaken (0.3) and still get (0.1), to say nothing of (0.2). It is also possible to construct a similar example of an entire function with preassigned order ρ, $0 \leqslant \rho \leqslant \infty$.

In §2 we construct an example of an entire function of order ρ, $1/2 \leqslant \rho \leqslant \infty$, for which 0 is an asymptotic value, but there exists no asymptotic curve Γ with the property (0.1) on which the function approaches 0. Since an entire function with $\rho < 1/2$ cannot have finite asymptotic values (see [4], p. 226), we have thereby obtained a negative answer to the question of Erdős. However, it is possible that (0.1) is always satisfied for some asymptotic curve on which the entire function f approaches a finite value if f is of order $\rho = 1/2$ and of normal type (if f is of minimal type, then it has no finite asymptotic values). For entire functions f of order $\rho = 1/2$ and normal type it is known only (see [5]) that for each asymptotic curve Γ on which f approaches a finite value,

$$\arg z = o\{(\ln |z|)^{1/2}\}, \quad z \to \infty, \quad z \in \Gamma,$$

where $\arg z$ is some branch of the argument which is continuous on Γ.

In §§1 and 2 we use various methods which can be used to construct examples of entire functions with other properties. We treat this briefly in §3. In particular, we obtain the following result in §3. We denote by $\theta(r, f)$ the measure of the set $\{\varphi \in [0, 2\pi]: |f(re^{i\varphi})| > 1\}$, where f is an entire function. Valiron ([6], pp. 133–136) has shown that for an entire function for which (0.3) is fulfilled, we have that $\theta(r, f) \to 2\pi$ as $r \to \infty$. Hayman [3] has strengthened this result by showing, in particular, that $\theta(r, f) = 2\pi$ for all r except for a set of finite logarithmic measure. In §3 we construct an example of an entire function f for which (0.4) is fulfilled for an arbitrary preassigned function $\varphi(r) \to \infty$ and for which $\theta(r, f) \to 0$ as $r \to \infty$, $r \in E \subset [0, \infty)$, where E is some set of upper density 1. Therefore the hypotheses under which the results of Valiron and Hayman mentioned above are true cannot be weakened. It is possible to construct the same examples with an arbitrary preassigned order. We note that the first examples of entire functions f of finite order for which $\underline{\lim}_{r\to\infty} \theta(r, f) = 0$ were constructed recently [7]. On the other hand, it is well known (see [8], [9], and [4], p. 233) that for entire functions of order ρ,

$$\overline{\lim_{r\to\infty}}\, \theta(r, f) \geqslant \min\{2\pi, \pi/\rho\}.$$

In addition we give in §3 an answer to a question of Winkler (see [2], Problem 2.42).

§1

THEOREM 1. *Suppose that the function $\varphi(r)$, defined on $[0, \infty)$, tends to $+\infty$ as $r \to \infty$. There exists an entire function f for which* (0.4) *holds but for which* (0.1) *is satisfied for no asymptotic curve Γ on which f tends toward ∞.*

PROOF. We denote by Γ_k the curve $\Gamma_k = \{z = re^{i\theta}: 2 \leqslant r \leqslant 3, \theta = 2\pi k(r - 2)\}$. We use the well-known theorem of Runge (see [10], Volume 1, Chapter 4, §2) to construct a polynomial P_k with the properties that

$$P_k(0) = 1, \tag{1.1}$$

$$P_k(z) \neq 0 \text{ for } |z| \leqslant 1, \tag{1.2}$$

$$|P_k(z)| \leqslant e^{-1} \text{ for } z \in \Gamma_k. \tag{1.3}$$

We put $x^\wedge = \max\{1, x\}$. It is clear that there exist positive constants A_k such that for all $r > 0$

$$\ln M(r, P_k) \leq A_k (\ln r)^\wedge, \quad k \in \mathbf{N}. \tag{1.4}$$

Without loss of generality, we may assume that $\varphi(r)$ is a continuously differentiable increasing function, $\varphi(r) \equiv 1$ for $r \in [0, e]$, $\varphi(r) \leqslant (\ln r)^\wedge$ and $\varphi'(r) \leqslant 1/r$, $0 \leqslant r < \infty$ (if this last condition is not fulfilled, we replace $\varphi(r)$ for $r \geqslant e$ by the function $\max\{1, \int_1^r \min\{\varphi'(t), 1/t\} dt\}$).

We construct a sequence of positive numbers (τ_n), $\tau_1 \leqslant e^{-1}$, $\tau_{n+1} \leqslant \tau_n/3$, and a sequence of polynomials (Q_n) with the following properties ($T_n = \tau_n^{-1}$):

$$\ln |Q_n(z)| \leq -2^{-1} - 2^{-n} \quad \text{for } z\tau_j \in \Gamma_j, \quad 1 \leq j \leq n, \tag{1.5}$$

$$N(r, 0, Q_n) \leq \varphi(r)(\ln^2 r)^\wedge \quad \text{for } r \geqslant 0, \tag{1.6}$$

$$\ln M(T_j, Q_n) \leq (1 - 2^{-n}) \varphi(T_j) \ln^2 T_j, \quad 1 \leq j \leq n. \tag{1.7}$$

We choose τ_1, $0 < \tau_1 < e^{-1}$, so that

$$\ln M(\tau_1 r, P_1) \leq \frac{1}{2} \varphi(r)(\ln^2 r)^\wedge, \quad r \geqslant 0. \tag{1.8}$$

We put $Q_1(z) = P_1(\tau_1 z)$. The polynomial Q_1 has the properties (1.5)–(1.7). Indeed, (1.5) (with $n = 1$) is satisfied for Q_1 by virtue of (1.3). From (1.1), (1.8) and Jensen's inequality it follows that

$$N(r, 0, Q_1) \leq \ln M(r, Q_1) \leq \frac{1}{2} \varphi(r)(\ln^2 r)^\wedge,$$

from which we get that (1.6) and (1.7) are satisfied for Q_1. We assume that $\tau_1, \ldots, \tau_{k-1}$ and $Q_1, \ldots, Q_{k-1}$ have already been constructed, satisfying (1.5)–(1.7). We show how to choose τ_k and Q_k. We choose $B_{k-1} > 1$ so that, for all $r > 0$,

$$\ln M(r, Q_{k-1}) \leq B_{k-1} (\ln r)^\wedge. \tag{1.9}$$

Let r_k be large enough so that

$$(1 - 2^{-k}) \varphi(r_k) \geqslant (6A_k + 1) B_{k-1}, \tag{1.10}$$

$$r_k \geqslant 3T_{k-1}. \tag{1.11}$$

Now we choose $\tau_k > 0$ small enough so that

$$\tau_k \leq 1/r_k. \tag{1.12}$$

$$2B_{k-1} \ln(3T_k) |\ln P_k(\tau_k z)| \leq 2^{-k}, \quad |z| \leq r_k, \tag{1.13}$$

where the branch of the logarithm $\ln P_k(\tau_k z)$ is chosen so that $\ln P_k(0) = 0$, taking into account the fact that $P_k(\tau_k z)$ has no zeros for $|z| \leqslant r_k$ by virtue of (1.2) and (1.12). To show that we can satisfy (1.13) we note that for $|z| \leqslant r_k$ and for sufficiently small τ

$$|\ln P_k(\tau z)| \ln \frac{3}{\tau} \leq 2 |P_k(\tau z) - 1| \ln \frac{3}{\tau}$$

$$\leq 2(1 + |P_k'(0)|) r_k \tau \ln \frac{3}{\tau} = o(1), \quad \tau \to 0.$$

Now we put

$$q_k = [2B_{k-1} \ln (3T_k)], \tag{1.14}$$

$$Q_k(z) = Q_{k-1}(z) \{P_k(\tau_k z)\}^{q_k}. \tag{1.15}$$

From (1.11) and (1.12) it follows that $\tau_k \leqslant \tau_{k-1}/3$. We show that (1.5)–(1.7) (with $n = k$) are satisfied for Q_k.

By virtue of (1.5) (with $n = k - 1$), (1.11) and (1.13)–(1.15), we have for $z\tau_j \in \Gamma_j$, $j = 1, \dots, k - 1$,

$$\begin{aligned} \ln |Q_k(z)| &= \ln |Q_{k-1}(z)| + q_k \ln |P_k(z\tau_k)| \\ &\leqslant -2^{-1} - 2^{-k+1} + 2^{-k} = -2^{-1} - 2^{-k}. \end{aligned} \tag{1.16}$$

By virtue of (1.3) and (1.9), for $z\tau_k \in \Gamma_k$ we have

$$\ln |Q_k(z)| \leqslant \ln |Q_{k-1}(z)| - q_k \leqslant B_{k-1} \ln (3T_k) - [2B_{k-1} \ln (3T_k)] \leqslant -1,$$

i.e., (1.16) is satisfied for $j = k$; consequently (1.5) holds for Q_k.

By virtue of (1.2), (1.6) with $n = k - 1$, and (1.15), for $r \leqslant T_k$ we get

$$N(r, 0, Q_k) = N(r, 0, Q_{k-1}) \leqslant \varphi(r) (\ln^2 r)^{\wedge}. \tag{1.17}$$

Using (1.4), (1.9), (1.10), (1.15) and (1.12), for $r \geqslant T_k \geqslant e$ we get

$$\begin{aligned} \ln M(r, Q_k) &\leqslant B_{k-1} \ln r + q_k A_k (\ln (\tau_k r))^{\wedge} \\ &\leqslant B_{k-1} \ln r + 2A_k B_{k-1} \ln (3T_k) \ln r \\ &\leqslant B_{k-1} \ln r + 4A_k B_{k-1} \ln r + 2A_k B_{k-1} \ln^2 r \leqslant B_{k-1} (1 + 6A_k) \ln^2 r \\ &\leqslant (1 - 2^{-k}) \varphi(r_k) \ln^2 r \leqslant (1 - 2^{-k}) \varphi(r) \ln^2 r. \end{aligned} \tag{1.18}$$

Taking (1.1), (1.17), (1.18) and Jensen's inequality into account, we see that (1.6) is also satisfied for $n = k$. Putting $r = T_k$ in (1.18), we get (1.7) with $n = k$ and $j = k$. If $1 \leqslant j \leqslant k - 1$, then from (1.7) for $n = k - 1$, (1.13), and (1.15) we have

$$\begin{aligned} \ln M(T_j, Q_k) &\leqslant \ln M(T_j, Q_{k-1}) + 2B_{k-1} \ln (3T_k) \ln M(T_j \tau_k, P_k) \\ &\leqslant (1 - 2^{-k+1}) \varphi(T_j) \ln^2 T_j + 2^{-k} \leqslant (1 - 2^{-k}) \varphi(T_j) \ln^2 T_j. \end{aligned} \tag{1.19}$$

Thus we have shown that (1.7) is satisfied for $n = k$. This proves the possibility of constructing (τ_n) and (Q_n) with the properties (1.5)–(1.7).

We show that (Q_n) converges uniformly on compact sets in **C** to some entire function f. The convergence of (Q_n) is equivalent to the convergence of the product

$$\prod_{k=2}^{\infty} \{Q_k(z)/Q_{k-1}(z)\} = \prod_{k=2}^{\infty} \{P_k(\tau_k z)\}^{q_k}$$

(see (1.15)), and this product converges uniformly on compact sets by virtue of (1.13) and (1.14).

We show that the entire function f satisfies the hypotheses of Theorem 1.

From (1.1) and (1.15) it follows that $Q_k(0) = 1$, $k \in \mathbf{N}$. Consequently $f(0) = 1$. From (1.5) we get that

$$\ln |f(z)| \leqslant -1/2, \qquad z\tau_j \in \Gamma_j, \qquad j \in \mathbf{N}. \tag{1.20}$$

From $f(0) = 1$ and (1.20) it follows, in particular, that $f \not\equiv$ const. Let Γ be some curve on which f approaches ∞. Taking the form of Γ_k into consideration, we get from (1.20) that

$$l(3T_k, \Gamma) \geqslant l(3T_k, \Gamma) - l(2T_k, \Gamma) \geqslant (k-1)\, 4\pi T_k.$$

Consequently (0.1) is not satisfied for Γ.

We estimate the growth of f. Allowing n to approach ∞ in (1.6) and using a well-known theorem of Hurwitz, we get that

$$N(r, 0, f) \leqslant \varphi(r)(\ln^2 r)^\wedge \leqslant (\ln^3 r)^\wedge, \quad r > 0. \tag{1.21}$$

Let $\pi(z)$ be a canonical Weierstrass product of genus zero constructed from the zeros of f. From (1.21) it follows that $\pi(z)$ has order zero. From (1.7) it follows that

$$\ln M(T_j, f) \leqslant \varphi(T_j) \ln^2 T_j \leqslant \ln^3 T_j.$$

Consequently $T(T_j, f/\pi) = O(\ln^3 T_j)$, $j \to \infty$, and $f/\pi \equiv$ const. (see, for example, [4], p. 51). Since $f(0) = \pi(0) = 1$, it follows that $f \equiv \pi$. Therefore ([4], p. 89, (4.16))

$$\ln M(r, f) \leqslant \int_0^\infty \ln\left(1 + \frac{r}{t}\right) dn(t, 0, f) \leqslant r \int_r^\infty \frac{N(t, 0, f)}{t^2}\, dt. \tag{1.22}$$

Since $\varphi'(r) \leqslant 1/r$, it follows that for all $r \geqslant r_0 \geqslant e$ with $\varphi(r_0) > 2$ the function $\varphi(r) r^{-1/2}$ is decreasing, and we get from (1.21) and (1.22) that

$$\ln M(r, f) \leqslant r \int_r^\infty \varphi(t)\, t^{-2} \ln^2 t\, dt$$

$$\leqslant \varphi(r) r^{1/2} \int_r^\infty t^{-3/2} \ln^2 t\, dt = 2(1 + o(1))\, \varphi(r) \ln^2 r, \quad r \to \infty.$$

Thus (0.4) is satisfied for f. This proves Theorem 1.

THEOREM 2. *Suppose that* $0 \leqslant \rho \leqslant \infty$. *There exists an entire function* f *of order* ρ *such that* (0.1) *is satisfied for no asymptotic curve* Γ *on which* f *approaches* ∞.

PROOF. For $\rho = 0$ Theorem 2 is contained in Theorem 1. Suppose that $0 < \rho < 1$. We define the sequences of curves (Γ_k), polynomials (P_k), and constants (A_k) as in the proof of Theorem 1. We take an entire transcendental function ψ such that

$$\ln M(r, \psi) = \ln|\psi(r)| \leqslant (r^\rho)^\wedge, \quad r \geqslant 0, \tag{1.23}$$

$$\psi(0) = 1, \quad \psi(z) \neq 0 \text{ for } |z| \leqslant 1, \tag{1.24}$$

$$\ln|\psi(r)| \geqslant \mu r^\rho \quad \text{for } r \geqslant 1, \tag{1.25}$$

where μ is some positive constant. It is possible to take $\psi(z)$ to be $E_\rho(\sigma z)$, for example, where $E_\rho(z)$ is the Mittag-Leffler function (see [4], p. 111), and $\sigma > 0$ is sufficiently small.

We construct sequences of positive numbers (τ_n), $\tau_1 \leqslant e^{-1}$, $\tau_{n+1} \leqslant \tau_n/3$, and (s_n), $s_1 \geqslant 1$, $s_{n+1} \geqslant 2s_n$, and a sequence of entire functions (R_n) with the following properties

$(T_n = 1/\tau_n)$:

$$\ln|R_n(z)| \leqslant -2^{-1}-2^{-n-1} \quad \text{for } z\tau_j \in \Gamma_j, \quad 1 \leqslant j \leqslant n, \tag{1.26}$$

$$N(r, 0, R_n) \leqslant 6(r^\rho \ln^2 r)^\wedge \quad \text{for } r \geqslant 0, \tag{1.27}$$

$$\ln|R_n(s_j)| \geqslant s_j^{(1-2^{-j})\rho} + 2^{-n}, \quad 1 \leqslant j \leqslant n, \tag{1.28}$$

$$\ln M(T_j, R_n) \leqslant 6(1-2^{-n}) T_j^\rho \ln^2 T_j, \quad 1 \leqslant j \leqslant n, \tag{1.29}$$

$$\ln M(r, R_n) = O(r^\rho), \quad r \to \infty. \tag{1.30}$$

We choose τ_1, $0 < \tau_1 \leqslant e^{-1}$, small enough so that

$$\ln M(\tau_1 r, P_1) \leqslant r^\rho, \qquad r \geqslant 1. \tag{1.31}$$

We now find σ_1, $\tau_1/3 > \sigma_1 > 0$, so small that

$$|\ln \psi(\sigma_1 z)| < \frac{1}{4} \quad \text{for } |z| \leqslant 3T_1. \tag{1.32}$$

Then, by virtue of (1.23) and the fact that $\sigma_1 < 1$,

$$\ln M(\sigma_1 r, \psi) \leqslant r^\rho, \quad r \geqslant 1. \tag{1.33}$$

We put $R_1(z) = P_1(\tau_1 z)\psi(\sigma_1 z)$. For $z\tau_1 \in \Gamma_1$ we have, by virtue of (1.3) and (1.32),

$$\ln|R_1(z)| \leqslant -1 + \ln M(\sigma_1 3T_1, \psi) \leqslant -1 + \frac{1}{4} = -2^{-1} - 2^{-2},$$

i.e., (1.26) is satisfied for R_1. From (1.31) and (1.33) it follows that

$$\ln M(r, R_1) \leqslant 2r^\rho, \quad r \geqslant 1,$$

whence it follows at once that (1.27), (1.29), and (1.30) hold. Since for $s_1 \geqslant \sigma_1^{-1}$

$$\ln|R_1(s_1)| \geqslant \mu\sigma_1^\rho s_1^\rho + O(\ln s_1), \quad s_1 \to \infty,$$

by virtue of (1.25), it follows that if we take $s_1 > 0$ large enough, (1.28) is satisfied.

We assume that $\tau_1, \ldots, \tau_{k-1}$, $s_1, \ldots, s_{k-1}$ and $R_1, \ldots, R_{k-1}$ have already been chosen so that (1.26)–(1.30) are satisfied, where R_j, $1 \leqslant j \leqslant k-1$, has the form

$$R_j(z) = \Omega_j(z) \prod_{\nu=1}^{j} \psi(\sigma_\nu z), \tag{1.34}$$

where $\Omega_j(z)$ is some polynomial, and where $0 < \sigma_\nu < 1$ for $1 \leqslant \nu \leqslant k-1$. We choose $C_{k-1} > 2$ large enough so that for all $r \geqslant 0$,

$$\ln M(r, R_{k-1}) \leqslant C_{k-1} r^\rho, \quad r \geqslant 1, \tag{1.35}$$

which is possible by virtue of (1.30) (for $n = k-1$). We take r_k so large that

$$r_k \geqslant \max\{3T_{k-1}, s_{k-1}\}, \tag{1.36}$$

$$\ln r_k \geqslant C_{k-1} + 2\cdot 3^\rho A_k C_{k-1} + 1. \tag{1.37}$$

Now we choose $\tau_k > 0$ small enough so that (1.12) is satisfied and

$$2C_{k-1}(3T_k)^\rho |\ln P_k(\tau_k z)| \leqslant 2^{-k-2}, \quad |z| \leqslant r_k. \tag{1.38}$$

We can satisfy (1.38) by choosing τ_k properly, since

$$\tau^{-\rho} |\ln P_k(\tau z)| \leqslant 2(1 + |P_k'(0)|) r_k \tau^{1-\rho}$$

for sufficiently small τ. We choose σ_k, $1/r_k > \sigma_k > 0$, so that

$$|\ln \psi(\sigma_k z)| \leqslant 2^{-k-2} \text{ for } |z| \leqslant 3T_k. \tag{1.39}$$

Now we put

$$R_k(z) = R_{k-1}(z)\{P_k(\tau_k z)\}^{p_k}\psi(\sigma_k z), \tag{1.40}$$

where

$$p_k = [2C_{k-1}(3T_k)^\rho]. \tag{1.41}$$

From (1.12) and (1.36) it follows that $\tau_k \leqslant \tau_{k-1}/3$. For $z\tau_j \in \Gamma_j$, $1 \leqslant j \leqslant k-1$, we have that

$$\begin{aligned} \ln|R_k(z)| &= \ln|R_{k-1}(z)| + p_k \ln|P_k(\tau_k z)| + \ln|\psi(\sigma_k z)| \\ &\leqslant -2^{-1} - 2^{-k} + 2^{-k-2} + 2^{-k-2} = -2^{-1} - 2^{-k-1} \end{aligned} \tag{1.42}$$

by virtue of (1.40), (1.26), (1.38), and (1.39). With the help of (1.35) and (1.39)–(1.41) we get that

$$\begin{aligned} \ln|R_k(z)| &\leqslant C_{k-1}(3T_k)^\rho - [2C_{k-1}(3T_k)^\rho] + 2^{-k-2} \\ &\leqslant -2 + 2^{-k-2} < -2^{-1} - 2^{-k-1} \end{aligned} \tag{1.43}$$

for $z\tau_k \in \Gamma_k$. Thus, by virtue of (1.42) and (1.43), (1.26) is satisfied for R_k with $n = k$.

For $1 \leqslant j \leqslant k-1$ we have

$$\begin{aligned} \ln|R_k(s_j)| &\geqslant \ln|R_{k-1}(s_j)| - p_k|\ln|P_k(\tau_k s_j)|| \\ &+ \ln|\psi(\sigma_k s_j)| \geqslant s_j^{(1-2^{-j})\rho} + 2^{-k+1} - 2^{-k-2} \geqslant s_j^{(1-2^{-j})\rho} + 2^{-k}. \end{aligned} \tag{1.44}$$

We have used (1.28) with $n = k-1$ and (1.38) here, as well as the fact that $|\psi(r)| \geqslant 1$ for $r \geqslant 0$. From (1.34) and (1.25) we get that, for sufficiently large s_k ($\geqslant 2s_{k-1}$)

$$\begin{aligned} \ln|R_k(s_k)| &\geqslant \ln|\Omega_j(s_k)| + \ln|\psi(\sigma_k s_k)| \\ &= O(\ln s_k) + \mu\sigma_k^\rho s_k^\rho \geqslant s_k^{(1-2^{-k})\rho} + 2^{-k}, \end{aligned}$$

which, together with (1.44), shows that (1.28) holds for $n = k$.

The function $\{P_k(\tau_k z)\}^{p_k}\psi(\sigma_k z)$ does not vanish for $|z| \leqslant T_k$; therefore for $r \leqslant T_k$

$$N(r, 0, R_k) = N(r, 0, R_{k-1}) \leqslant 6(r^\rho \ln^2 r)^\wedge, \tag{1.45}$$

by virtue of (1.27) for $n = k-1$. With the help of (1.40), (1.35), (1.4), (1.23), (1.12), and (1.37) we get that

$$\begin{aligned} \ln M(r, R_k) &\leqslant C_{k-1}r^\rho + p_k A_k(\ln(\tau_k r))^\wedge + (\sigma_k^\rho r^\rho)^\wedge \\ &\leqslant C_{k-1}r^\rho + 2A_k C_{k-1}(3T_k)^\rho \ln r + r^\rho \leqslant (C_{k-1} + 2\cdot 3^\rho A_k C_{k-1} + 1)r^\rho \ln r \\ &\leqslant (\ln T_k) r^\rho \ln r \leqslant r^\rho \ln^2 r \end{aligned} \tag{1.46}$$

for $r \geqslant T_k$. From (1.45), (1.46), and Jensen's inequality, we get (1.27) for $n = k$. Putting $r = T_k$ in (1.46), we get (1.29) for $n = k$ and $j = k$. If $1 \leqslant j \leqslant k-1$, then by virtue of

(1.29) with $n = k - 1$ and (1.38) we get

$$\ln M(T_j, R_k) \leqslant \ln M(T_j, R_{k-1}) + 2C_{k-1}(3T_k)^\rho \ln M(T_j\tau_k, P_k) + \ln M(\sigma_k T_j, \psi)$$
$$\leqslant 6(1-2^{-k+1}) T_j^\rho \ln^2 T_j + 2^{-k-2} + 2^{-k-2} \leqslant 6(1-2^{-k}) T_j^\rho \ln^2 T_j,$$

so that (1.29) with $n = k$ is satisfied for $j \leqslant k - 1$.

The relation (1.30) for $n = k$ follows immediately from (1.34).

Thus we have constructed the desired sequences (τ_n), (s_n), and (R_n).

The uniform convergence on compact sets in $\mathbf{C}$ of the sequence (R_n) of entire functions to the entire function f is proved in exactly the same way as the convergence of (Q_n) in the proof of Theorem 1 (with reference to (1.38) and (1.39)). Allowing n to approach ∞ in (1.26)–(1.29), we get that (1.20) is satisfied for f and that

$$N(r, 0, f) \leqslant 6(r^\rho \ln^2 r)^\wedge, \quad r \geqslant 0, \tag{1.47}$$

$$\ln|f(s_j)| \geqslant s_j^{(1-2^{-j})\rho}, \quad j \in \mathbf{N}, \tag{1.48}$$

$$\ln M(T_j, f) \leqslant 6T_j^\rho \ln^2 T_j, \quad j \in \mathbf{N}. \tag{1.49}$$

As we saw in the proof of Theorem 1, it follows from (1.20) that (0.1) does not hold for f. From (1.47) and (1.49) we can conclude that f is a canonical product of genus zero whose order is the same as that of $N(r, 0, f)$ by the classical Borel theorem (see [4], p. 79), and whose order does not exceed ρ by virtue of (1.47). From (1.48) it follows that the order of f is equal to ρ. Thus the entire function f has all of the required properties.

Considering entire functions of the form $f(z^n)$, $n = 2, 3, \ldots,$ where f is the function of order ρ, $0 < \rho < 1$, constructed above, we show that Theorem 2 holds for arbitrary ρ, $0 < \rho < \infty$.

If $\rho = \infty$, then we carry out the construction as follows. We take $\gamma = \{z = re^{ir}: r \geqslant 1\}$ and, using Carleman's approximation theorem (see [11], Chapter II, §§1 and 2), we construct an entire function f which is bounded on γ. Clearly if Γ is an arbitrary asymptotic curve on which f approaches ∞, then $r^2 = O(l(r, \Gamma))$ as $r \to \infty$, and for a continuous branch of $\arg z$ on Γ we have $\arg z = |z| + O(1)$ as $z \to \infty$, $z \in \Gamma$. By Ahlfors' theorem (see [12] and the footnote on p. 213 of the Russian translation of [13]), f has infinite order.

§2

The method by which we will construct the necessary example of an entire function with finite asymptotic values is different from the method of §1, which is not applicable here. The basis of our construction is a theorem on conformal mapping of a semistrip which may be of interest in itself.

Let A and B be two open sets. We call the set $C = \operatorname{int}(\overline{A \cup B})$, where the bar denotes closure and int denotes the interior, their *connection* $C = A \amalg B$. The operation of connection generalizes in a natural way to the case of an arbitrary system of open sets. This operation is studied in topology (see [13], Russian p. 14), but we know of no name or notation for it.

We denote by L_m the linear function $L_m(z) = z + m$, $m \in \mathbf{R}$, and we define $I = \{iy: y \leqslant \pi\}$ and put $I_m = L_m(I)$.

Let $0 < m < \infty$. We denote by $D(m)$ an arbitrary Jordan region in $\mathbf{C}$ having the following properties:

1) $I \subset \partial D(m)$, $I_m \subset \partial D(m)$;

2) $\bar{D}(m) \cap \{z: x < 0, |y| \leqslant \pi\} = \varnothing$, $\bar{D}(m) \cap \{z: x > m, |y| \leqslant \pi\} = \varnothing$.

We will regard $D(m)$ as a Jordan quadrangle with vertices $\pm \pi i$ and $m \pm \pi i$, and we will call I and I_m the *lateral sides* of $D(m)$, and each of the two Jordan arcs which constitute $\partial D(m) \setminus (I \cup I_m)$ a *base* of $D(m)$. In the special case when $D(m)$ is a rectangle with vertices $\pm \pi i$ and $m \pm \pi i$, we will write $Q(m)$ instead of $D(m)$. We will denote by $S\{n_j, D_j(m_j)\}$ (where $n_j > 0$ and $m_j > 0$ are real numbers) the curvilinear semistrip

$$S = S\{n_j, D_j(m_j)\} = Q(n_1) \amalg L_{d_1}(D_1(m_1)) \amalg L_{\delta_2}(Q(n_2)) \amalg L_{d_2}(D_2(m_2)) \amalg$$
$$\ldots \amalg L_{\delta_k}(Q(n_k)) \amalg L_{d_k}(D_k(m_k)) \amalg L_{\delta_{k+1}}(Q(n_{k+1})) \amalg \ldots,$$

where $d_1 = n_1$, $d_k = \delta_k + n_k$ and $\delta_k = d_{k-1} + m_{k-1}$, $k = 2, 3, \ldots$.

In order to facilitate the description of this and the semistrips and strips which we will encounter below, we assume that the n_j are chosen large enough so that the regions $L_{d_j}(D_j(m_j))$, $j \in \mathbf{N}$, are pairwise disjoint. Without this assumption, we would have to regard the common points of the various $L_{d_j}(D_j(m_j))$ as distinct and the semistrip $S(n_j, D_j(m_j))$ as a Riemann surface. None of the following propositions would be weakened by such an approach, but the statements would be more complicated.

Let $S_0 = \{z: x > 0, |y| < \pi\}$, and let $\zeta = \zeta_S(z)$ be a one-to-one conformal mapping of S_0 onto S such that $\zeta_S(\pm i\pi) = \pm i\pi$ and $\zeta_S(\infty) = \infty$. We will denote the inverse of $\zeta = \zeta_S(z)$ by $z = z_S(\zeta)$.

In what follows, whenever we consider a one-to-one conformal mapping of one Jordan region onto another, we will assume that it has been extended by continuity to the boundary.

THEOREM 3. *For any sequence of regions $(D_j(m_j))$, $j \in \mathbf{N}$, it is possible to find a sequence of positive numbers (ν_j) such that for any semistrip $S\{n_j, D_j(m_j)\}$ with $n_j \geqslant \nu_j$,*

$$\lim_{x \to +\infty} \operatorname{Re} \zeta_S(x + iy)/x = 1, \tag{2.1}$$

where the approach to 1 is uniform with respect to y.

We will need the following lemma.

LEMMA 1. *Let $z = z(\zeta)$ be a one-to-one conformal mapping of the region*

$$\Delta = \Delta(n, D(m)) = Q(n) \amalg L_n(D(m)) \amalg L_{n+m}(Q(n))$$

onto some rectangle $Q(M)$, $M = M_\Delta$, such that $z(\pm i\pi) = \pm i\pi$ and $z(2n + m \pm i\pi) = M \pm i\pi$. Let $\zeta = \zeta(z) = \xi(z) + i\eta(z)$ be the inverse of $z = z(\zeta)$. Then for each $\varepsilon > 0$ and $m > 0$ there exists an n such that for $|y| \leqslant \pi$

$$|\eta(iy) - y| < \varepsilon, \quad |\eta(M + iy) - y| < \varepsilon, \tag{2.2}$$

$$\left|\frac{d\eta(iy)}{dy} - 1\right| < \varepsilon, \quad \left|\frac{d\eta(M + iy)}{dy} - 1\right| < \varepsilon. \tag{2.3}$$

PROOF. For a set $E \subset \mathbf{C}$ and a line or segment $l \subset \mathbf{C}$, we will denote by E/l the set which is symmetric to E with respect to l. We assume that $n > 16\pi$. Then by a well-known inequality of Ahlfors (see [15], Chapter V, §6, Theorem 6.1), $\operatorname{Re} z(\zeta) \geqslant n - 8\pi$ for $\zeta \in I_n$. Consequently $J_n = \zeta(I_{n-8\pi})$ is a Jordan arc lying in $Q(n)$ (with the exception of its endpoints) and connecting the bases of $Q(n)$. Again applying Ahlfors' inequality,([1]) we get that $\operatorname{Re}\zeta \geqslant n - 16\pi$ for $\zeta \in J_n$. We put $Q_0(n) = \zeta(Q(n-8\pi))$ and note that

$$Q(n-16\pi) \subset Q_0(n) \subset Q(n). \tag{2.4}$$

Similarly, appealing to Ahlfors' inequality, we get that for $8\pi < x < n - 8\pi$,

$$Q(x-8\pi) \subset \zeta(Q(x)) \subset Q(x+8\pi). \tag{2.5}$$

We put

$$Q_0'(n) = Q_0(n) \amalg (Q_0(n)/I),$$

$$\Omega(n) = Q_0'(n) \amalg (Q_0'(n)/\{\zeta : \eta = -\pi\}) \amalg (Q_0'(n)/\{\zeta : \eta = \pi\}),$$

$$Q'(n) = Q(n-8\pi) \amalg (Q(n-8\pi)/I),$$

$$\omega(n) = Q'(n) \amalg (Q'(n)/\{\zeta : \eta = -\pi\}) \amalg (Q'(n)/\{\zeta : \eta = \pi\}),$$

i.e., $\omega(n)$ is a rectangle with vertices $\pm(n - 8\pi) \pm 3\pi i$.

From the Riemann-Schwarz symmetry principle, it follows easily that the mapping $\zeta = \zeta(z)$ can be continued analytically to a one-to-one conformal mapping of $\omega(n)$ onto $\Omega(n)$. We take an arbitrary sequence (n_j) of real numbers which approaches $+\infty$ monotonically and for which $n_1 > 16\pi$. We let $\omega_j = \omega(n_j)$ and $\Omega_j = \Omega(n_j)$, and we let $\zeta_j = \zeta_j(z)$ be the corresponding mapping of ω_j onto Ω_j; it is the analytic continuation of $\zeta = \zeta(z)$ of the rectangle $Q(M_\Delta)$ onto $\Delta = \Delta(n_j, D(m))$.

If we take (2.4) into account, it is easy to see that both sequences of regions, (ω_j) and (Ω_j), converge to the same kernels in the sense of Carathéodory (see [10], Volume 2, Chapter 5, §2.3), the strips $\omega(\infty) = \{z: |y| < 3\pi\}$ and $\Omega(\infty) = \{\zeta: |\eta| < 3\pi\}$. Since $\Omega_j \subset \Omega(\infty)$, it follows by an application of the compactness principle that we can extract from $(\zeta_j(z))$ a subsequence $(\zeta_{j_k}(z))$ which converges uniformly on compact sets in $\omega(\infty)$ to a function $\zeta_\infty(z)$ which is analytic in $\omega(\infty)$ (the fact that each of the functions $\zeta_j(z)$ is defined only on a subset of $\omega(\infty)$ does not preclude our conclusion; see [10], Volume 2, Chapter 5, §2.3, proof of Carathéodory's theorem). By virtue of well-known theorems $\zeta_\infty(z)$ maps $\omega(\infty)$ conformally and univalently onto some region $\Omega_1(\infty) \subset \Omega(\infty)$. On the other hand, it is easy to prove using (2.5) that $\zeta_\infty(S_0) = S_0$, and since $\zeta_\infty(+\infty) = +\infty$ and $\zeta_\infty(\pm i\pi) = \pm i\pi$, it follows that $\zeta_\infty(z) \equiv z$. Consequently $\Omega_1(\infty) = \Omega(\infty)$, and the sequence $(\zeta_{j_k}(z))$ converges to $\zeta_\infty(z) \equiv z$ uniformly on compact sets. In particular we have that

$$\operatorname{Im}\zeta_{j_k}(iy) \rightrightarrows y; \quad \frac{d\operatorname{Im}\zeta_{j_k}(iy)}{dy} \rightrightarrows 1 \quad \text{for } k \to \infty,$$

on I, since $\zeta_j(iy) = i\operatorname{Im}\zeta_j(iy)$. Taking k sufficiently large, we can satisfy the first inequalities in (2.2) and (2.3). In order to satisfy the second inequalities in (2.2) and (2.3), we must use the fact that the intervals I and I_M are completely equal in Lemma 1 and, if necessary, choose an additional subsequence from $(\zeta_{j_k}(z))$.

REMARK. Following the pattern of the proof of a well-known theorem of Carathéodory (see [10], Volume 2, Chapter 5, §2.3), it is possible to show that any sequence $(\zeta_j(z))$ converges uniformly on compact sets to the identity mapping. Hence it is possible to deduce, in turn, that under the hypotheses of the lemma, (2.2) and (2.3) are satisfied for all sufficiently large n. It is also clear that (2.2) follows from (2.3).

([1])Ahlfors' inequality is usually stated for a conformal mapping of a curvilinear semistrip onto a rectilinear semistrip, but an examination of the proof shows that it is applicable in both of our cases.

LEMMA 2. *Let $\eta = \lambda(y)$ be a continuously differentiable function on $[-\pi, \pi]$ which maps this interval bijectively onto $[-\pi, \pi]$, and let*

$$\lambda(\pm\pi) = \pm\pi, \quad |\lambda(y) - y| < \varepsilon, \quad |\lambda'(y) - 1| < \varepsilon, \quad |y| \leqslant \pi,$$

where $0 < \varepsilon \leqslant 1/2$. Then the quasiconformal mapping of $Q(1)$ onto $Q(1)$ given by

$$\zeta^+(z) = \begin{cases} \xi = x, \\ \eta = (\lambda(y) - y)x + y \end{cases} \tag{2.6}$$

or

$$\zeta^-(z) = \begin{cases} \xi = x, \\ \eta = (y - \lambda(y))x + \lambda(y), \end{cases} \tag{2.7}$$

has characteristic $p(z)$ such that

$$p(z) - 1 \leqslant \varepsilon(1 + \sqrt{5}), \quad z \in Q(1). \tag{2.8}$$

PROOF. It is well known (see, for example, [4], pp. 437–440) that

$$p(z) = K + \sqrt{K^2 - 1},$$

where

$$K = (E+G)/2J, \quad E = (\xi'_x)^2 + (\eta'_x)^2, \quad G = (\xi'_y)^2 + (\eta'_y)^2, \quad J = \xi'_x\eta'_y - \xi'_y\eta'_x.$$

We consider the mapping (2.6), for example. For it,

$$J = (\lambda'(y) - 1)x + 1 \geqslant \min\{1, \ \lambda'(y)\} \geqslant 1 - \varepsilon \geqslant 1/2,$$
$$E + G = 1 + (\lambda(y) - y)^2 + \{(\lambda'(y) - 1)x + 1\}^2,$$
$$K - 1 = \{(\lambda'(y) - 1)^2x^2 + (\lambda(y) - y)^2\}/2J \leqslant 2\varepsilon^2.$$

Hence we get

$$p - 1 = \sqrt{K-1}\,(\sqrt{K-1} + \sqrt{K+1}) \leqslant \varepsilon\sqrt{2}\,(\sqrt{2\varepsilon^2} + \sqrt{2\varepsilon^2 + 2})$$
$$\leqslant \varepsilon(1 + \sqrt{5}).$$

We pass directly to the proof of Theorem 3. According to Lemma 1 we can find $n_j^0 > 0$ such that the function $z = z_j(\zeta)$ maps $\Delta(n_j^0, D_j(m_j))$ conformally and univalently onto the rectangle $Q(M_j)$, $z(\pm i\pi) = \pm i\pi$, $z(2n_j^0 + m_j \pm i\pi) = M_j \pm i\pi$, and its inverse $\zeta = \zeta_j(z)$ satisfies (2.2) and (2.3) with $\varepsilon = (j+1)^{-2}$, $\eta(z) = \eta_j(z)$ and $M = M_j$, $j \in \mathbf{N}$. For (ν_j) we take an arbitrary sequence which approaches ∞ and which satisfies the conditions

1) $\nu_j > 2 + n_{j-1}^0 + n_j^0, \quad j \geqslant 2, \quad \nu_1 \geqslant n_1^0 + 1,$

2) $$n_j^0 = o(\nu_j), \quad j \to \infty, \tag{2.9}$$

3) $$m_j = o(\nu_j), \quad j \to \infty, \tag{2.10}$$

4) $$M_j = o(\nu_j), \quad j \to \infty. \tag{2.11}$$

We show that (ν_j) is the required sequence.

First of all, we construct a quasiconformal mapping $w = w_S(z)$ of the semistrip S_0 onto $S = S\{n_j, D_j(m_j)\}$, where $n_j \geqslant \nu_j$. To do this, we represent S_0 and S as connections of pairwise disjoint Jordan quadrangles:

$$S = Q(n_1 - n_1^0) \amalg L_{p_1}(\Delta(n_1^0, D_1(m_1))) \amalg L_{q_2}(Q(n_2 - n_1^0 - n_2^0))$$
$$\amalg L_{p_2}(\Delta(n_2^0, D_2(m_2))) \amalg \cdots \amalg L_{q_k}(Q(n_k - n_{k-1}^0 - n_k^0))$$
$$\amalg L_{p_k}(\Delta(n_k^0, D_k(m_k))) \amalg L_{q_{k+1}}(Q(n_{k+1} - n_k^0 - n_{k+1}^0)) \amalg \cdots,$$

where $p_1 = n_1 - n_1^0$, $p_k = q_k + n_k - n_{k-1}^0 - n_k^0$, $q_k = p_{k-1} + m_k + 2n_k^0$, $k = 2, 3, \ldots,$ and

$$S_0 = Q(n_1 - n_1^0) \amalg L_{p_1'}(Q(M_1)) \amalg L_{q_2'}(Q(n_2 - n_1^0 - n_2^0))$$
$$\amalg L_{p_2'}(Q(M_2)) \amalg \cdots \amalg L_{q_k'}(Q(n_k - n_{k-1}^0 - n_k^0)) \amalg L_{p_k'}(Q(M_k)) \amalg \ldots,$$

where $p_1^1 = n_1 - n_1^0$, $p_k' = q_k' + n_k - n_{k-1}^0 - n_k^0$, $q_k' = p_{k-1}' + M_k$, $k = 2, 3, \ldots.$

We put

$$w_S(z) = L_{p_k}(\zeta_k(L_{-p_k'}(z))) \quad \text{for } z \in L_{p_k'}(Q(M_k)), \quad k \in \mathbf{N}. \tag{2.12}$$

Thus

$$w_S(L_{p_k'}(Q(M_k))) = L_{p_k}(\Delta(n_k^0, D_k(m_k))),$$

and the mapping $w_S(z)$ is conformal and univalent in $L_{p_k'}(Q(M_k))$, $k \in \mathbf{N}$. For $z \in L_{q_k'}(\overline{Q}(n_k - n_{k-1}^0 - n_k^0))$, $k = 2, 3, \ldots,$ we define $w_S(z)$ in the following way:

$$w_S(z) = \begin{cases} L_{q_k}(\zeta_k^-(L_{-q_k'}(z))), & z \in L_{q_k'}(\overline{Q}(1)), \\ L_{q_k - q_k'}(z), & z \in L_{q_k'+1}(\overline{Q}(n_k - n_{k-1}^0 - n_k^0 - 2)), \\ L_{p_k-1}(\zeta_k^+(L_{-p_k'+1}(z))), & z \in L_{p_k'-1}(\overline{Q}(1)), \end{cases} \tag{2.13}$$

where $\zeta_k^-(z)$ is defined in accordance with (2.7), where we take $\lambda(y)$ to be $\eta_{k-1}(M_{k-1} + iy)$, and $\zeta_k^+(z)$ is defined in accordance with (2.6), where we take $\lambda(y)$ to be $\eta_k(iy)$. It is not difficult to verify that the $w_S(z)$ defined in (2.13) is continuous in

$$L_{q_k'}(Q(n_k - n_{k-1}^0 - n_k^0))$$

and maps this rectangle onto

$$L_{q_k}(Q(n_k - n_{k-1}^0 - n_k^0)),$$

and the mapping is conformal in

$$L_{q_k'+1}(Q(n_k - n_{k-1}^0 - n_k^0 - 2))$$

and quasiformal in

$$L_{q_k'}(\overline{Q}(1)) \text{ and } L_{p_k'-1}(\overline{Q}(1)).$$

The characteristic of the quasiconformal mapping in these rectangles satisfies the relations

$$p(z)-1\leqslant k^{-2}(1+\sqrt{5}),\quad z\in L_{q_k'}(\overline{Q}(1)),\tag{2.14}$$

$$p(z)-1\leqslant (k+1)^{-2}(1+\sqrt{5}),\quad z\in L_{p_k'-1}(\overline{Q}(1)),\tag{2.15}$$

which can be seen if we take into account (2.8) and the choice of ε in the definition of $\zeta_j(z)$. For $z\in\overline{Q}(n_1-n_1^0)$ we define $w_S(z)$ thus:

$$w_S(z)=\begin{cases} z, & z\in\overline{Q}(n_1-n_1^0-1),\\ L_{p_1-1}(\zeta_1^+(L_{-p_1'+1}(z))), & z\in L_{p_1'-1}(\overline{Q}(1)),\end{cases}\tag{2.16}$$

where $\zeta_1^+(z)$ is defined in accordance with (2.6), where we take $\lambda(y)$ to be $\lambda(y)=\eta_1(iy)$. As before, we get that (2.15) is satisfied for $k=1$.

The function $w=w_S(z)$ defined by (2.12), (2.13), and (2.16) maps the semistrip S_0 conformally onto the semistrip $S=S\{n_j, D_j(m_j)\}$ in the w-plane, and by virtue of (2.14) and (2.15),

$$\iint_{S_0}\{p(z)-1\}\,dx\,dy\leqslant 4\pi(1+\sqrt{5})\sum_{k=1}^{\infty}(k+1)^{-2}<\infty.\tag{2.17}$$

Taking (2.9), (2.10), and (2.11) into account, we get that for $k\to\infty$

$$p_k=\sum_{j=1}^{k}n_j+\sum_{j=1}^{k-1}m_j-n_k^0=(1+o(1))\sum_{j=1}^{k}n_j,\tag{2.18}$$

$$q_{k+1}=\sum_{j=1}^{k}n_j+\sum_{j=1}^{k}m_j+n_k^0=(1+o(1))\sum_{j=1}^{k}n_j,\tag{2.19}$$

$$p_k'=\sum_{j=1}^{k}n_j+\sum_{j=1}^{k-1}M_j-2\sum_{j=1}^{k-1}n_j^0-n_k^0=(1+o(1))\sum_{j=1}^{k}n_j,\tag{2.20}$$

$$q_{k+1}'=\sum_{j=1}^{k}n_j+\sum_{j=1}^{k}M_j-2\sum_{j=1}^{k}n_j^0+n_k^0=(1+o(1))\sum_{j=1}^{k}n_j.\tag{2.21}$$

From (2.12) it follows that for $p_k'\leqslant x\leqslant q_{k+1}'$ we have

$$p_k\leqslant \operatorname{Re} w_S(x+iy)\leqslant q_{k+1}.$$

From (2.18)–(2.21) we get that

$$\operatorname{Re} w_S(x+iy)/x\to 1\ \text{ for } x\to\infty,\quad x\in\bigcup_{k=1}^{\infty}[p_k, q_{k+1}],\tag{2.22}$$

uniformly in y. But if $q_k'\leqslant x\leqslant p_k'$, then by virtue of (2.13), (2.19), and (2.21) we have

$$\operatorname{Re} w_S(x+iy)=q_k-q_k'+x=o(q_k')+x=(1+o(1))x,\quad k\to\infty.\tag{2.23}$$

From (2.22) and (2.23) it follows that

$$\lim_{x\to+\infty}\operatorname{Re} w_S(x+iy)/x=1,\tag{2.24}$$

where the limit is uniform with respect to y.

Since the mapping $\zeta_S\colon S_0 \to S$ is conformal, it follows that $\zeta_S^{-1}(w_S(z))$ maps S_0 quasiconformally onto itself; in addition, the points $\pm i\pi$ and ∞ are invariant under this transformation, and (2.17) is satisfied. According to a simple consequence (see [16], Lemma 2) of a well-known theorem of Teichmüller and Belinskiĭ ([17], Theorem 12),

$$\begin{aligned} \zeta_S^{-1}(w_S(x+iy)) &= x+iy+\lambda+o(1), \quad x \to +\infty, \\ w_S^{-1}(\zeta_S(x+iy)) &= x+iy-\lambda+o(1), \quad x \to +\infty, \end{aligned} \tag{2.25}$$

where λ is some real number and $o(1)$ approaches 0 uniformly with respect to y. From (2.25) and (2.24) we get that

$$\begin{aligned} \zeta_S(x+iy) &= w_S(x+iy-\lambda+o(1)), \quad x \to +\infty, \\ \operatorname{Re}\zeta_S(x+iy) &= \operatorname{Re} w_S(x+iy-\lambda+o(1)) = (1+o(1))(x-\lambda+o(1)) \\ &= (1+o(1))x, \quad x \to +\infty, \end{aligned} \tag{2.26}$$

i.e., (2.1).

COROLLARY. *Under the hypotheses of Theorem* 3,

$$\operatorname{Re} z_S(\zeta) = (1+o(1))\operatorname{Re}\zeta, \quad \operatorname{Re}\zeta \to +\infty,\ \zeta \in S, \tag{2.27}$$

where $o(1)$ *approaches* 0 *uniformly with respect to* $\eta = \operatorname{Im}\zeta$, $\zeta \in S$.

LEMMA 3. *Let a semistrip* $S = S\{n_j, D_j(m_j)\}$ *which intersects each line* $\{z\colon \operatorname{Re} z = \text{const} > 0\}$ *in an interval of length* $\leqslant 2\pi$ *be given. Let* α *and* β *be any numbers such that* $-\pi < \alpha < \beta < \pi$. *Denote the image of the region* $S_0(\alpha, \beta) = S_0 \cap \{z\colon \alpha < y < \beta\}$ *under the mapping* $w = \exp\zeta_S(z)$ *by* $E(\alpha, \beta)$. *Then there exists a sequence* (ν_j^0) *of positive numbers such that, for any semistrip* S *with* $n_j \geqslant \nu_j^0$, *except* (2.1), *we have the following property. There exists in* $E(\alpha, \beta)$ *a locally rectifiable curve* C *connecting a point on the circle* $\{w\colon |w| = 1\}$ *with* ∞ *such that*:

1) $$\int_C |w|^{-2}|dw| < \infty, \tag{2.28}$$

2) *there exists an* R *such that, for any* $w_0 \in C$, $|w_0| \geqslant R$, *we have that* $\{w\colon |w - w_0| \leqslant 1\} \subset E(\alpha, \beta)$.

PROOF. Let $\alpha < \alpha_1 < \beta_1 < \beta$, and let $E^1(\alpha_1, \beta_1)$ be the image in the w-plane of $S_0(\alpha_1, \beta_1)$ under the mapping $w = \exp w_S(z)$, where the quasiconformal mapping $w_S\colon S_0 \to S$ is defined as in the proof of Theorem 3. We also retain here all of the other notation introduced in the proof of Theorem 3. We denote the image of the segment

$$\left\{x + i\frac{\alpha_1+\beta_1}{2}\colon\ 0 \leqslant x \leqslant M_j\right\}$$

under the mapping $\zeta = \zeta_j(z)$ by γ_j, and the image of $\{z\colon 0 \leqslant x \leqslant M_j, y = \alpha_1, \beta_1\}$ under the same mapping by γ_j'. Let d_j' be the distance from γ_j to γ_j', let

$$\kappa_j = \frac{1}{2}\min\{1, d_j'(\beta_1 - \alpha_1)/2\},$$

and let $l(\gamma_j)$ be the length of γ_j. We choose $\nu_j^0 \geqslant \nu_j$ so large that for any $n_j \geqslant \nu_j^0$

$$l(\gamma_j) \leqslant j^{-2}\exp p_j, \quad j \in \mathbf{N}, \tag{2.29}$$

$$\varkappa_j \exp p_j > 4e^2, \quad j \in \mathbf{N}. \tag{2.30}$$

It is obvious that (2.1) is satisfied for the semistrip S. We show how to construct a curve C with the properties 1) and 2). Let

$$J=\{x+i(\alpha_1+\beta_1)/2:\ 0\leqslant x<\infty\};$$

let C be the image of J under the mapping $w = \exp w_S(z)$, and let J' be the image of J under the mapping $\zeta = w_S(z)$. If

$$\zeta_0 \in J' \cap \{L_{p_j}(\Delta(n_j^0, D_j(m_j))) \amalg L_{p_j-1}(Q(1)) \amalg L_{q_j}(Q(1))\},$$

then, as is not difficult to see,

$$\{\zeta : |\operatorname{Re}(\zeta-\zeta_0)|<\varkappa_j,\ |\operatorname{Im}(\zeta-\zeta_0)|<\varkappa_j\} \subset S(\alpha_1, \beta_1), \tag{2.31}$$

where $S(\alpha_1, \beta_1)$ is the image of $S_0(\alpha_1, \beta_1)$ under the mapping $\zeta = w_S(z)$. Let $w_0 = \exp \zeta_0$; then

$$\begin{aligned}\bar{U}(1, w_0) = \{w : |w-w_0|\leqslant 1\} \subset \{w : |w_0|e^{-\varkappa_j}<|w|<|w_0|e^{\varkappa_j},\\ |\arg w-\arg w_0|<\varkappa_j\} \subset E^1(\alpha_1, \beta_1).\end{aligned} \tag{2.32}$$

The first of the inclusions in (2.32) follows from (2.30), since

$$|w_0|e^{\varkappa_j}-|w_0|>|w_0|-|w_0|e^{-\varkappa_j}\geqslant e^{p_j-1}(1-e^{-\varkappa_j})>e^{p_j-2}\varkappa_j>4$$

and

$$|w_0|e^{-\varkappa_j}\varkappa_j\geqslant e^{p_j-2}\varkappa_j>4.$$

The second inclusion in (2.32) follows from (2.31). If

$$\zeta_0 \in J' \cap L_{q_j+1}(Q(n_j-n_{j-1}^0-n_j^0-2)),\quad j\geqslant 2,$$

then

$$\left\{\zeta : |\operatorname{Re}(\zeta-\zeta_0)|<\frac{1}{2},\ |\operatorname{Im}(\zeta-\zeta_0)|<\frac{\beta_1-\alpha_1}{4}\right\} \subset S(\alpha_1, \beta_1),$$

and $\bar{U}(1, w_0) \subset E^1(\alpha_1, \beta_1)$ for all sufficiently large j. Thus for all $w_0 \in C$, $|w_0| > R_1$, we have $\bar{U}(1, w_0) \subset E^1(\alpha_1, \beta_1)$. By virtue of (2.26) there exists $R > R_1 + 1$ such that

$$\{w:\ |w|\geqslant R-1\}\cap E^1(\alpha_1, \beta_1) \subset E(\alpha, \beta),$$

so 2) holds for C.

We show that (2.28) is fulfilled. We have

$$\begin{aligned}\int_C |w|^{-2}|dw| = \sum_{j=1}^{\infty} \int_{C\cap[e^{p_j}, e^{q_j+1}]} |w|^{-2}|dw|\\ + \int_{C\cap[1, e^{p_1}]} |w|^{-2}|dw| + \sum_{j=2}^{\infty} \int_{C\cap[e^{q_j}, e^{p_j}]} |w|^{-2}|dw|.\end{aligned} \tag{2.33}$$

It is easy to see that

$$\sum_{j=2}^{\infty} \int_{C\cap[e^{q_j}, e^{p_j}]} |w|^{-2}|dw| \leqslant \text{const} \int_1^{\infty} |w|^{-2}\, d|w| < \infty. \tag{2.34}$$

On the other hand,

$$\begin{aligned}\int\limits_{C\cap[e^{p_j},e^{q_{j+1}}]} |w|^{-2}|dw| &\geqslant e^{-p_j}\int\limits_{C\cap[e^{p_j},e^{q_{j+1}}]} |d\ln w| \\ &= e^{-p_j}\int\limits_{J'\cap L_{p_j}(\Delta(n_j^0,D_j(m_j)))} |d\zeta| = e^{-p_j}l(\gamma_j)\leqslant\frac{1}{j^2},\end{aligned}\tag{2.35}$$

by virtue of (2.29). Then (2.28) follows from (2.33), (2.34), and (2.35).

In general, the curve C constructed in this way is contained in $E(\alpha, \beta)$ only from some point on. It is clear how to modify its definition so that 1) and 2) are satisfied and $C \subset E(\alpha, \beta)$. This proves Lemma 3.

We now prove the principal theorem of this section.

THEOREM 4. *For each ρ, $1/2 \leqslant \rho \leqslant \infty$, there exists an entire function f of order ρ for which* 0 *is an asymptotic value, but there exists no asymptotic curve Γ with the property* (0.1) *on which f approaches* 0.

PROOF. Let $1/2 < \rho < \infty$, and let ε be a number such that

$$0<\varepsilon<\min\left\{\frac{\pi}{2}\left(1-\frac{1}{2\rho}\right), \frac{\pi}{2\rho}\right\}.$$

We construct a semistrip $S = S\{n_j, D_j(m_j)\}$ for which

$$D_j(m_j) = D_j(1) = D_j^- \amalg D_j^+,$$

where D_j^- is a parallelogram with vertices at $\pm\pi i$ and $1/2 + (2j\pi \pm \pi)i$, $D_j^+ = D_j^-/\{z\colon x = 1/2\}$, and the sequence n_j is chosen so that (2.1) is fulfilled and so that Lemma 3 holds both with

$$\alpha' = \frac{\pi}{2\rho}+\varepsilon,\quad \beta' = \min\left\{\pi, \frac{3\pi}{2\rho}\right\}-\varepsilon,$$

and with

$$\alpha'' = -\min\left\{\pi, \frac{3\pi}{2\rho}\right\}+\varepsilon,\quad \beta'' = -\frac{\pi}{2\rho}-\varepsilon.$$

We denote the image of S_0 under the mapping $w = \exp\zeta_S(z)$ by E. It is easy to see that E is the region $\{w\colon 1 < |w| < \infty\}$ with a cut along some curve L which connects $w = -1$ with $w = \infty$. Let $\varphi(w) = z_S(\ln w)$, where we choose a single-valued branch of $\ln w$ such that $\ln 1 = 0$. We draw curves C' and C'' in E which have the properties of the curve C in Lemma 3 and which correspond to the choice $\alpha = \alpha'$, $\beta = \beta'$ and the choice $\alpha = \alpha''$, $\beta = \beta''$, respectively. Let C_0 be the arc of the unit circle which connects the ends of the curves C' and C'' lying on the unit circle and which does not go through $w = -1$. The curve $C'' + C_0 + C'$ divides the finite w-plane into two regions; we denote the one which contains $w = 0$ by E^- and the other one by E^+.

We put $\Phi(w) = \exp\exp(\rho\varphi(w))$. If $w \in E(\alpha', \beta') \cup E(\alpha'', \beta'')$, then

$$\frac{\pi}{2\rho}+\varepsilon<|\operatorname{Im}\varphi(w)|<\frac{3\pi}{2\rho}-\varepsilon.\tag{2.36}$$

From (2.27) it follows that

$$\operatorname{Re}\varphi(w) = (1+o(1))\ln|w|, \quad w\to\infty, \quad w\in E. \tag{2.37}$$

From (2.36) and (2.37) it follows that for $w \in E(\alpha', \beta') \cup E(\alpha'', \beta'')$, and all the more so for $w \in \partial E^-$,

$$|\Phi(w)| \leqslant \exp\{-\sin(\rho\varepsilon)|w|^{(1+o(1))\rho}\}, \quad w\to\infty. \tag{2.38}$$

Taking (2.28) into account, we get that the Cauchy integral

$$\frac{1}{2\pi i}\int_{\partial E^-}\frac{\Phi(\tau)}{\tau - w}\,d\tau = \begin{cases} f_-(w), & w\in E^-,\\ f_+(w), & w\in E^+,\end{cases} \tag{2.39}$$

converges absolutely for $w \notin \partial E^-$ and defines analytic functions $f_-(w)$ in E^- and $f_+(w)$ in E^+. It is possible to show by standard methods ([18], Chapter II, §1, Example 4; [4], pp. 242–243) that $f_-(w)$ can be continued analytically as an entire function f to $\mathbf{C}$, and

$$f(w) = \begin{cases} f_-(w), & w\in E^-,\\ f_+(w)+\Phi(w), & w\in E^+.\end{cases} \tag{2.40}$$

We show that

$$f_-(w) = O\left(\frac{1}{w}\right), \quad w\to\infty, \ w\in E^-, \tag{2.41}$$

$$f_+(w) = O\left(\frac{1}{w}\right), \quad w\to\infty, \ w\in E^+. \tag{2.42}$$

From (2.28) and (2.38) it follows that

$$\frac{1}{2\pi}\int_{\partial E^-}|\Phi(\tau)||d\tau| \leqslant \frac{1}{2\pi}\int_{\partial E^-}|\tau||\Phi(\tau)||d\tau| = K < \infty. \tag{2.43}$$

We write ($w \in E_-$)

$$f_-(w) = -\frac{1}{w}\frac{1}{2\pi i}\int_{\partial E^-}\Phi(\tau)\,d\tau + \frac{1}{w}\frac{1}{2\pi i}\int_{\partial E^-}\frac{\tau\Phi(\tau)}{\tau - w}\,d\tau. \tag{2.44}$$

From (2.43) it follows that

$$\left|\frac{1}{2\pi i}\int_{\partial E^-}\Phi(\tau)\,d\tau\right| \leqslant K. \tag{2.45}$$

If $w \in E^-$ and $\operatorname{dist}(w, \partial E^-) \geqslant 1/2$, then we get easily from (2.43) that

$$\left|\frac{1}{2\pi i}\int_{\partial E^-}\frac{\tau\Phi(\tau)}{\tau - w}\,d\tau\right| \leqslant 2K. \tag{2.46}$$

If $w \in E^-$ and $\operatorname{dist}(w, \partial E^-) < 1/2$, then

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial E^-}\frac{\tau\Phi(\tau)}{\tau - w}\,d\tau &= \frac{1}{2\pi i}\int_{-\partial(E^+\setminus C(w))}\frac{\tau\Phi(\tau)}{\tau - w}\,d\tau \\ &+ \frac{1}{2\pi i}\int_{-\partial(E^+\cap C(w))}\frac{\tau\Phi(\tau)}{\tau - w}\,d\tau = \frac{1}{2\pi i}\int_{-\partial(E^+\setminus C(w))}\frac{\tau\Phi(\tau)}{\tau - w}\,d\tau,\end{aligned} \tag{2.47}$$

where $C(w) = \{\tau: |\tau - w| \leqslant 1/2\}$. If $|w| > R$, where R is defined as in Lemma 3, then $C(w) \subset E(\alpha', \beta') \cup E(\alpha'', \beta'')$. From (2.43), (2.38), and (2.47) we get that for $\operatorname{dist}(w, \partial E^-) < 1/2$, $|w| > R$,

$$\left| \frac{1}{2\pi i} \int_{\partial E^-} \frac{\tau\Phi(\tau)}{\tau - w}\, d\tau \right| \leqslant 2K + 2\max\{|\tau\Phi(\tau)| : \tau \in C(w)\} = 2K + o(1), \quad w \to \infty. \tag{2.48}$$

Then (2.41) follows from (2.44), (2.45), (2.46), and (2.48). The proof of (2.42) is similar. From (2.40), (2.41), and (2.42) it follows that

$$f(w) = \begin{cases} O\left(\frac{1}{w}\right), & w \to \infty, \ w \notin E^+, \\ O\left(\frac{1}{w}\right) + \Phi(w), & w \to \infty, \ w \in E^+. \end{cases} \tag{2.49}$$

It follows immediately from (2.49) that 0 is an asymptotic value of f.

If J is the curve $w = w(t) = \exp \zeta_S(t)$, $0 \leqslant t < \infty$, then $\varphi(w(t)) \equiv t$, and $\Phi(w(t)) = \exp\exp(\rho t)$. Since $J \subset E$, if we take into account our choice of the regions $D_j(1)$ and the form of S and E, which depends on it, we arrive at the conclusion that, for any asymptotic curve Γ on which f approaches 0,

$$l(e^{d_{j+1}}, \Gamma) \geqslant l(e^{d_{j+1}}, \Gamma) - l(e^{d_j}, \Gamma) \geqslant 2\pi e^{d_j} 2(j-1), \quad j \geqslant j_0,$$

whence it follows immediately that (0.1) is not fulfilled for Γ.

It remains to determine the order of f. It is clear that

$$\begin{aligned} \ln\ln M(r, f) &\leqslant O(1) + \ln\ln\max\{|\Phi(w)| : |w| = r, w \in E^+\} \\ &\leqslant O(1) + \rho\max\{\operatorname{Re}\varphi(w) : |w| = r, w \in E^+\} \\ &= O(1) + \rho\max\{\operatorname{Re} z_S(\zeta) : \operatorname{Re}\zeta = \ln r, \zeta \in S\}. \end{aligned}$$

By virtue of (2.27) we have

$$\ln\ln M(r, f) \leqslant (1 + o(1))\rho \ln r, \quad r \to \infty. \tag{2.50}$$

On the other hand, if $w = w(t)$, $0 \leqslant t < \infty$, is a parametric representation of J, then

$$\begin{aligned} \ln\ln|f(w(t))| &= O(1) + \ln\ln|\Phi(w(t))| = \rho t + O(1) \\ &= (1 + o(1))\rho \ln|w(t)|, \quad t \to \infty, \end{aligned} \tag{2.51}$$

since $\ln|w(t)| = \operatorname{Re}\zeta_S(t) = (1 + o(t))t$ as $t \to \infty$, by virtue of (2.1). From (2.50) and (2.51) it follows that

$$\ln\ln M(r, f) \sim \rho \ln r, \quad r \to \infty, \tag{2.52}$$

i.e., the order and the lower order of f are equal to ρ. Thus we have constructed the required example for $1/2 < \rho < \infty$.

We pass to the case $\rho = 1/2$. We will need certain more complex arguments which would have been valid also for $1/2 < \rho < \infty$ with small additions. In order to avoid burdening the proof, we will use certain geometrically obvious facts without formal proof.

We take a semistrip $S = S\{n_j, D_j(m_j)\}$ for which

$$D_j(m_j) = D_j(1) = D_j^- \amalg Q_j \amalg D_j^+,$$

where D_j^- is a parallelogram with vertices at $\pm\pi i$ and $1/3 + (2j\pi \pm \pi)i$, $D_j^+ = D_j^-/\{z: x = 1/2\}$, and Q_j is a rectangle with vertices at $1/3 + (2\pi j \pm \pi)i$ and $2/3 + (2\pi j \pm \pi)i$. We define the set E, the curve L, and the analytic function $\varphi(w) = z_S(\ln w)$ in E as above. Let the sequence (n_j) be chosen so that (2.1) is satisfied and

$$l(r, L) = O(r^2), \quad r \to \infty. \tag{2.53}$$

The function $\Phi_1(w) = \exp\exp\{\varphi(w)/2\}$ is analytic in E,

$$\ln|\Phi_1(w)| = \exp\left\{\frac{1}{2}\operatorname{Re} z_S(\ln w)\right\}\cos\left\{\frac{1}{2}\operatorname{Im} z_S(\ln w)\right\} > 0$$

in E, and $\ln|\Phi_1(w)| = 0$ on L.

We put $U(r, w_0) = \{w: |w - w_0| < r\}$. We choose the points $b_2^+, b^+, b_1^+, -1, b_1^-, b^-$, and b_2^- on the arc $\{w: |w| = 1, \operatorname{Re} w < 0\}$ so that they are encountered in that order as we traverse the arc in a counterclockwise direction, and we let $C_2^+, C^+, C_1^+, L, C_1^-, C^-$, and C_2^- be pairwise disjoint curves in $\mathbf{C}$ which connect these points, respectively, with $w = \infty$. The curve L is determined by S, and the curves $C_2^+, C^+, C_1^+, C_1^-, C^-$, and C_2^- can be drawn so that

$$\operatorname{dist}(w, L) < 0.5|w|^{-2}, \quad w \in C_2^{\pm}, \tag{2.54}$$

$$\operatorname{dist}(w, C_j^{\pm}) > 0,1|w|^{-2}, \quad w \in C^{\pm}, \ |w| \geqslant R_0 > 3, \ j = 1, 2, \tag{2.55}$$

$$l(r, C^{\pm}) = O(r^2), \quad r \to \infty, \tag{2.56}$$

$$\operatorname{length}(C_j^{\pm} \cap U(2, w_0)) < 8, \quad w_0 \in C^{\pm}, \ j = 1, 2. \tag{2.57}$$

We denote by G_2 (by G, by G_1) that region which contains L: $L \subset G_1 \subset G \subset G_2$, and which is bounded by the curves $C_2^{\pm}$ (the curves $C^{\pm}$, the curves $C_1^{\pm}$) and the shortest arc of the unit circle connecting $b_2^{\pm}$ ($b^{\pm}$, $b_1^{\pm}$).

From (2.1) it follows that $\ln|\Phi_1(w)| \leqslant |w|^{1/2+o(1)}$ for $w \to \infty$, $w \in E$. Consequently there exists a positive constant A such that for any $w_0 \in L$, $|w_0| \geqslant 1$, and $w \in E \cap U(1, w_0)$

$$\operatorname{Re}\exp\left\{\frac{1}{2}\varphi(w)\right\} \leqslant A(|w_0| + 1)^{3/4}. \tag{2.58}$$

In our case it follows from (2.54) that

$$\operatorname{dist}(w, L) < 0.5|w|^{-2}, \quad w \in G_2. \tag{2.59}$$

If $w_0 \in L$ and $|w_0| \geqslant 2$, then the curve L divides the disc $U(1, w_0)$ into two regions in each of which the analytic function

$$\Psi(w) = \Phi_1(w)\exp\{-A(|w_0| + 1)^{3/4}\}$$

satisfies the inequality $|\Psi(w)| < 1$ (by virtue of (2.58)), and at boundary points of these regions which lie in $U(1, w_0)$ we have

$$|\Psi(w)| = \exp\{-A(|w_0| + 1)^{3/4}\}.$$

Therefore by a well-known theorem of Milloux (see [19], Chapter VIII, §4, Theorem 6),

$$\ln|\Psi(w)| \leqslant -\frac{2}{\pi}\arcsin\left(\frac{1 - |w - w_0|}{1 + |w - w_0|}\right)A(|w_0| + 1)^{3/4},$$

whence

$$\ln|\Phi_1(w)| \leqslant A(|w_0|+1)^{3/4}\left(1-\frac{2}{\pi}\arcsin\frac{1-|w-w_0|}{1+|w-w_0|}\right) \tag{2.60}$$

$$=\frac{2A}{\pi}(|w_0|+1)^{3/4}\operatorname{arc\,tg}\frac{2|w-w_0|^{1/2}}{1-|w-w_0|}\leqslant\frac{4A}{\pi}(|w_0|+1)^{3/4}\frac{|w-w_0|^{1/2}}{1-|w-w_0|}.$$

From (2.59) we get that for $w \in \overline{G}_2$ and $|w| > R_1$ there exists $w_0 \in L \cap U(1, w_0)$ such that $\operatorname{dist}(w, L) < |w_0|^{-2}$. Together with (2.60), this gives

$$\ln|\Phi_1(w)| \leqslant \frac{4A}{\pi}(|w_0|+1)^{3/4}|w_0|^{-1}(1-|w_0|^{-2})^{-1},$$

i.e., $\Phi_1(w)$ is bounded in $\overline{G}_2 \cap E$, and consequently

$$|\Phi_1(w)| \leqslant A_1, \quad w \in \overline{G}_2 \cap E, \tag{2.61}$$

on $C^{\pm}$, $C_1^{\pm}$, and $C_2^{\pm}$.

We put $G' = G \cup U(2, 0)$ and

$$\Phi_2(w) = \frac{1}{2\pi i}\int_{\partial G'}\frac{\Phi_1(\tau)}{\tau^4(\tau-w)}\,d\tau, \quad w \notin \partial G'.$$

This Cauchy integral converges absolutely, since

$$\frac{1}{2\pi}\int_{\partial G'}|\Phi_1(\tau)||\tau|^{-4}|d\tau| \leqslant \frac{A_1}{2\pi}\int_{\partial G'}|\tau|^{-4}|d\tau| < \infty \tag{2.62}$$

by virtue of (2.56). If we continue $\Phi_2(w)$ analytically from G' into $\mathbf{C}$, we get (see (2.40)) the entire function f:

$$f(w)=\begin{cases}\Phi_2(w), & w\in G',\\ \Phi_2(w)+\Phi_1(w)w^{-4}, & w\notin \overline{G}'.\end{cases}$$

It remains to show that

$$\Phi_2(w) = O\left(\frac{1}{w}\right), \quad w\to\infty, \ w\notin\partial G' \tag{2.63}$$

(see (2.41) and (2.42)); then we get

$$f(w)=\begin{cases}O\left(\dfrac{1}{w}\right), & w\to\infty, \ w\in\overline{G}',\\[2ex] O\left(\dfrac{1}{w}\right)+\Phi_1(w)w^{-4}, & w\to\infty, \ w\notin\overline{G}',\end{cases} \tag{2.64}$$

and the treatment of this example is concluded just as in the case $1/2 < \rho < \infty$, using (2.64) instead of (2.49). In order to prove (2.63), we write (see (2.44))

$$\Phi_2(w) = -\frac{1}{w}\frac{1}{2\pi i}\int_{\partial G'}\Phi_1(\tau)\tau^{-4}\,d\tau + \frac{1}{w}\frac{1}{2\pi i}\int_{\partial G'}\frac{\Phi_1(\tau)\tau^{-3}}{\tau-w}\,d\tau. \tag{2.65}$$

The boundedness of the integrals in (2.65) can be proved from (2.61) and (2.62) in roughly the same way as the boundedness of the integrals in (2.44). We consider, for example, the second integral in (2.65). If $\operatorname{dist}(w, \partial G') \geqslant 1$, then, by virtue of (2.56),

$$\left|\frac{1}{2\pi i}\int\limits_{\partial G'}\frac{\Phi_1(\tau)}{\tau^3(\tau-w)}\,d\tau\right| \leqslant \frac{A_1}{2\pi}\int\limits_{\partial G'}|\tau|^{-3}|d\tau| = K < \infty. \tag{2.66}$$

If $\operatorname{dist}(w, \partial G') < 1$, $w \notin \overline{G}$ and $|w| > R_0 + 1$, then, letting

$$G''(w) = \{G' \setminus U(1, w)\} \cup G_1, \ \Gamma(w) = \partial G''(w) \cap \overline{U}(1, w),$$

we get

$$\left|\frac{1}{2\pi i}\int\limits_{\partial G'}\frac{\Phi_1(\tau)}{\tau^3(\tau-w)}\,d\tau\right| = \left|\frac{1}{2\pi i}\int\limits_{\partial G''(w)}\frac{\Phi_1(\tau)}{\tau^3(\tau-w)}\,d\tau\right| \leqslant K + \left|\frac{1}{2\pi i}\int\limits_{\Gamma(w)}\frac{\Phi_1(\tau)}{\tau^3(\tau-w)}\,d\tau\right|$$

$$\leqslant K + \frac{2\pi+16}{2\pi}A_1\frac{10(|w|+1)^2}{(|w|-1)^3} = K + o(1), \quad w\to\infty, \ w \notin \overline{G}.$$

Here we have used (2.66) as well as (2.55) and (2.57) with $j = 1$. If $\operatorname{dist}(w, \partial G') < 1$, $w \in \overline{G}$ and $|w| > R_0 + 1$, then instead of $G''(w)$ we choose

$$G'''(w) = (\mathbf{C} \setminus \overline{G}') \setminus (\overline{U}(1, w) \cap \overline{G}_2)$$

and rely on (2.55) and (2.57) with $j = 2$.

Thus we have constructed the required examples for $1/2 \leqslant \rho < \infty$. We note that in these examples (0.1) is also not satisfied for asymptotic curves on which f approaches ∞. By comparing with Theorem 2 (for $1/2 \leqslant \rho < \infty$) we get additional information: for the functions we have constructed, the lower order is equal to the order (see (2.52)).

For $\rho = \infty$, we argue in exactly the same way as in the corresponding case in the proof of Theorem 2, except that we choose the entire function f not only to be bounded but also to approach 0 on γ.

§3

For the definition of $\theta(r, f)$, see the Introduction. If $E \subset [0, \infty)$ is a locally measurable set, and mes E denotes the linear measure of E, then the upper density $D^*(E)$ is defined by

$$D^*(E) = \overline{\lim_{r\to\infty}}\frac{1}{r}\operatorname{mes}(E \cap [0, r]).$$

THEOREM 5. *For any function* $\varphi(r)$ *which approaches* $+\infty$ *as* $r \to +\infty$, *there exist an entire function* f *satisfying* (0.4) *and two subsets*

$$\mathbf{R}_+ : E' = \bigcup_{k=1}^{\infty}[a_k', b_k'], \quad E'' = \bigcup_{k=1}^{\infty}[a_k'', b_k''], \quad b_k'/a_k' \to \infty, \quad b_k''/a_k'' \to \infty,$$

such that

$$\theta(r, f) \to 0 \ \ \text{for} \ \ r\to\infty, \ r \in E' \cup E'', \tag{3.1}$$

$$|f(z)| < 1 \ \ \text{for} \ \ |z| \in E', \ \operatorname{Re} z \geqslant 0 \ \ \text{and for} \ \ |z| \in E'', \ \operatorname{Re} z \leqslant 0, \tag{3.2}$$

$$T(r, f) = o(\ln M(r, f)) \ \ \text{for} \ \ r\to\infty, \ r \in E' \cup E''. \tag{3.3}$$

There also exist an entire function f *of order* ρ, $0 \leqslant \rho < \infty$, *and sets* E' *and* E'' *of the form described above for which* (3.1)–(3.3) *are satisfied.*

The proof is the same as the proofs of Theorems 1 and 2, except that we take for the

Γ_k not curves, as in §1, but sets

$$\Gamma_{2n-1} = \{z : 2 \leqslant |z| \leqslant 2n+1,\ |\arg z| < n\pi/(n+1)\},$$
$$\Gamma_{2n} = \{z : 2 \leqslant |z| \leqslant 2n+2,\ |\arg z - \pi| < n\pi/(n+1)\}, \quad n \in \mathbf{N},$$

and instead of $\tau_{k+1} \leqslant \tau_k/3$ we require that $\tau_{k+1} \leqslant \tau_k/(k+2)$. The relation (2.3) follows from (3.1), since

$$T(r, f) \leqslant \theta(r, f) \ln^+ M(r, f), \quad r \geqslant 0.$$

For $1/2 \leqslant \rho < \infty$ it is possible to repeat the proof of Theorem 4 taking

$$D_j(m_j) = D_j(j+2) = Q(1) \amalg Q_j \amalg L_{j+1}(Q(1)),$$

where

$$Q_{2n-1} = \{z:\ 1 < x < 2n,\ -\pi < y < -n\pi/(n+1)\},$$
$$Q_{2n} = \{z:\ 1 < x < 2n+1,\ 0 < y < \pi/(n+1)\}.$$

In this case, (2.52) will be satisfied, in addition.

We did not mention the case $\rho = \infty$ in Theorem 5, since for $\rho = \infty$ it is easy to find appropriate examples, and even well-known examples, of entire functions with properties which are, in a certain sense, stronger; for example, $E(z + 2\pi i)$, where $E(z)$ is the Mittag-Leffler function (see [15], Chapter VI, §4) or the functions in [20] (Part IV, Chapter 3, §3).

We note that it follows from the hypotheses $a'_k = o(b'_k)$ and $a''_k = o(b''_k)$ as $k \to \infty$ in Theorem 5 that $D^*(E') = D^*(E'') = D^*(E' \cup E'') = 1$. As was mentioned in the Introduction, Theorem 5 improves one of the results of [7]. Theorem 5 also improves a result of Piranian [21] which asserts the existence of an entire function f with property (0.4) such that on every ray issuing from $z = 0$ there exists an infinite sequence of pairwise disjoint segments of length 1 on which $|f(z)| < 1$. Finally, Theorem 5 improves a result of Paley [22] which asserts the existence of an entire function of arbitrary order ρ, $0 \leqslant \rho \leqslant \infty$, for which (3.3) is satisfied for some sequence (r_k), $r_k \to \infty$.

None of the properties (3.1), (3.2) or (3.3) can hold for an entire function f which satisfies (0.3). This remark has already been made in the Introduction about (3.1). From a result of Valiron (see [6], pp. 133–136) it follows easily that for entire functions which satisfy (0.3) we always have that $T(r, f) \sim \ln M(r, f)$ as $r \to \infty$. A weaker property than (3.2) given in the example of Piranian is already inconsistent with (0.3). In fact, Hayman [3] has shown that it follows from (0.3) that the sets $\{r > 1 : |f(re^{i\theta})| < 1\}$ are bounded for almost all $\theta \in [0, 2\pi]$ and have finite logarithmic length for all $\theta \in [0, 2\pi]$.

In conclusion we discuss one more question related to the form of asymptotic curves. It is well known that there exist entire functions f of finite order for which it is possible to construct an asymptotic curve on which f approaches $a \neq \infty$ such that all a-points of f lie on this curve, and there exist entire functions such that on each asymptotic curve with asymptotic value a there lie at most a finite number of a-points. The canonical Weierstrass product

$$f(z) = \prod_{k=1}^{\infty} E(zk^{-1/\rho}, [\rho]), \quad \rho > [\rho],$$

provides an example of the first type if $\operatorname{tg} \rho\pi > 0$ and of the second type if $\operatorname{tg} \rho\pi < 0$ (see [23], Chapter II). The problems become considerably more complex if we take into account several asymptotic values simultaneously. We quote a problem of Winkler (see [2], Problem 2.42).

"Let $f(z)$ be an entire function (of sufficiently high order) with n ($\geqslant 2$) different asymptotic values a_k. Suppose that γ_k is a path such that $f(z) \to a_k$ ($z \to \infty$, $z \in \gamma_k$). Let $n(r, a_k, \gamma_k)$ be the number of zeros of $f(z) - a_k$ on γ_k and in $|z| \leqslant r$. Can we find a function $f(z)$ such that

$$\frac{n(r, a_k, \gamma_k)}{n(r, a_k)} \to b_k > 0 \quad \text{as} \quad r \to \infty, \tag{3.4}$$

for $k = 1, 2, \ldots, n$? Can we take $b_k = 1$?"

We give here an affirmative answer to the first question; the second remains open.

We consider the entire function

$$f(z) = \int_0^z \frac{\sin \zeta^n}{\zeta^n} \, d\zeta, \quad n = 2, 3, \ldots.$$

It is easy to verify that

$$f(ze^{i\frac{\pi k}{n}}) = e^{i\frac{\pi k}{n}} f(z), \quad k = 0, 1, \ldots, 2n - 1. \tag{3.5}$$

The functions $f(z)$ and $f'(z) = z^{-n} \sin z^n$ are of order ρ and of normal type. Let γ_k be the ray $\{r \exp(i\pi k/n) \colon 0 \leqslant r < \infty\}$, $0 \leqslant k \leqslant 2n - 1$. It is easy to see that the ray γ_0 is an asymptotic curve which corresponds to the asymptotic value

$$a_0 = \int_0^\infty x^{-n} \sin x^n \, dx = \frac{1}{n-1} \cos \frac{\pi}{2n} \Gamma\left(\frac{1}{n}\right)$$

(see, for example, [24], No. 710). From (3.5) we get that f approaches $a_k = a_0 e^{i\pi k/n}$, $k = 0, 1, \ldots, 2n - 1$, on the asymptotic curve γ_k. We fix the value of φ, $0 < \varphi < \pi/n$. Then

$$\begin{aligned} f(re^{i\varphi}) = {} & \frac{e^{-i(n-1)\varphi}}{2i} \int_0^r t^{-n} \{\exp(it^n \cos n\varphi - t^n \sin n\varphi) \\ & - \exp(-it^n \cos n\varphi + t^n \sin n\varphi)\} \, dt \\ = {} & O(1) - \frac{e^{-i(n-1)\varphi}}{2i} \int_1^r t^{-n} \exp(-it^n \cos n\varphi + t^n \sin n\varphi) \, dt \\ = {} & O(1) - \frac{e^{-i(n-1)\varphi}}{2in} \int_1^{r^n} \tau^{\frac{1}{n} - 2} \exp(-i\tau e^{in\varphi}) \, d\tau \\ = {} & -(1 + o(1)) \frac{1}{2n} (re^{i\varphi})^{1-2n} \exp(-ir^n e^{in\varphi}), \quad r \to \infty, \end{aligned}$$

where the last equality is obtained after integrating the preceding integral twice by parts. Hence we find that ($0 < \varphi < \pi/n$)

$$\ln |f(re^{i\varphi})| \sim r^n \sin n\varphi, \quad r \to \infty. \tag{3.6}$$

Hence f has completely regular growth in the angle $\{z\colon 0 < \arg z < \pi/n\}$ in the sense of Levin and Pfluger [23]. Using (3.5), (3.6) and the results of [23] (Chapter III, §1), we get that f is a function of completely regular growth with indicator $h(\varphi, f) = |\sin n\varphi|$, $\varphi \in [0, 2\pi]$. Since the indicator $h(\varphi, f)$ is positive on $[0, 2\pi]$ except at a finite number of points, it follows that $f - a_k$ has completely regular growth and that

$$h(\varphi, f-a_k) = |\sin n\varphi|, \quad \varphi \in [0, 2\pi], \quad k = 0, 1, \ldots, 2n-1. \tag{3.7}$$

Using a well-known formula (see [23], Chapter IV, §1, proof of Theorem 3) and (3.7), we get

$$\lim_{r\to\infty} \frac{n(r, a_k)}{r^n} = \frac{n}{2\pi}\int_0^{2\pi} h(\varphi, f-a_k)\, d\varphi = \frac{2n}{\pi}. \tag{3.8}$$

We put $x_j = (\pi j)^{1/n}$, $j = 0, 1, \ldots$. It is easy to see that

$$f(x_{2\nu}) < f(x_{2\nu+1}), \quad f(x_{2\nu+2}) < f(x_{2\nu+1}), \quad \nu = 0, 1, 2, \ldots.$$

Furthermore,

$$|f(x_{j+1}) - f(x_j)| = \int_{x_j}^{x_{j+1}} x^{-n} |\sin x^n|\, dx > \int_{x_j}^{x_{j+1}} \frac{x^{n-1}|\sin x^n|}{(x^n+\pi)^{2-1/n}}\, dx$$

$$= \int_{x_{j+1}}^{x_{j+2}} x^{-n} |\sin x^n|\, dx = |f(x_{j+2}) - f(x_{j+1})|.$$

Then

$$a_0 = \lim_{j\to\infty} f(x_j) = \sum_{j=0}^{\infty} \{f(x_{j+1}) - f(x_j)\},$$

where the alternating convergent series satisfies the hypotheses of Leibniz' theorem. Then $a_0 > f(x_{2\nu})$ and $a_0 < f(x_{2\nu+1})$, $\nu = 0, 1, \ldots$. Taking into account the fact that $f'(x) \neq 0$ in (x_j, x_{j+1}), we get that f has exactly one a_0-point of the first order in each interval (x_j, x_{j+1}), $j = 0, 1, 2, \ldots$. Now we easily find that $n(r, a_0, \gamma_0) \sim r^n/\pi$ as $r \to \infty$. Taking (3.5) into account, we get that

$$n(r, a_k, \gamma_k) \sim r^n/\pi, \quad k = 0, 1, \ldots, 2n-1. \tag{3.9}$$

Then (3.4) with $b_k = 1/2n$, $k = 0, 1, \ldots, 2n-1$, follows from (3.8) and (3.9).

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