

String with beads

Let us consider a massless string of length L stretched with force σ . Suppose that the ends of the string are fixed and n beads of mass m are placed with equal spacing

$$l = \frac{L}{n+1} \quad (1)$$

between them. Denoting by x_j the displacement of the j -th bead from the equilibrium, and assuming that displacements are small in comparison with l , we obtain the following system of equations:

$$m \frac{d^2 x_j}{dt^2} = \frac{\sigma}{l} ((x_{j+1} - x_j) - (x_j - x_{j-1})), \quad j = 1, \dots, n,$$

with the boundary conditions $x_0 = 0$ and $x_{n+1} = 0$. Introducing $x_j = e^{i\omega t} u_j$, we obtain

$$u_{j+1} - \left(2 - \frac{ml}{\sigma} \omega^2\right) u_j + u_{j-1} = 0$$

with the boundary conditions $u_0 = u_{n+1} = 0$.

Let us denote for convenience

$$2 - \frac{ml}{\sigma} \omega^2 = 2 \cos \theta, \quad (2)$$

where θ is real or complex. Then

$$u_{j+1} - 2u_j \cos \theta + u_{j-1} = 0.$$

This is a difference equation with constant coefficients. To find the general solution we plug $u_j = \rho^j$ and obtain the characteristic equation

$$\rho^2 - 2\rho \cos \theta + 1 = 0,$$

so $\rho = \exp \pm i\theta$. So our difference equation has general solution

$$u_j = c_1 \cos j\theta + c_2 \sin j\theta. \quad (3)$$

From the first boundary condition we obtain $c_1 = 0$ and from the second one $\sin(n+1)\theta = 0$, so

$$\theta_k = \frac{\pi k}{n+1}.$$

From (2) we obtain the frequencies

$$\omega_k = 2\sqrt{\frac{\sigma}{ml}} \sin \frac{\pi k}{2(n+1)}.$$

The eigenvectors (amplitudes) are obtained from (3) with $c_1 = 0$ and $c_2 = 1$:

$$u_{k,j} = \sin \frac{\pi j k}{n+1}.$$

We can pass to the limit when $n \rightarrow \infty$. Using (1) and $m = \rho L/n$ where ρ is the density of the string (the units are mass/length) we obtain in the limit

$$\omega_k = \sqrt{\frac{\sigma}{\rho}} \frac{\pi k}{L},$$

the law discovered by Marin Mersenne and published in his book “Harmonie universelle” in 1636. The book dealt with the theory of music and musical instruments.

Notice the following consequences of this law.

1. All frequencies of a homogeneous string are integer multiples of the fundamental one,

$$\omega_1 = \sqrt{\frac{\sigma}{\rho}} \frac{\pi}{L}.$$

This fundamental frequency is called the base tone, and the rest are overtones.

2. The base tone is inverse proportional to the length. This fact was known to the ancient Greeks, and traditionally it is credited to Pythagoras himself. Anyway, this is probably the earliest known result of mathematical physics.

3. Despite a lot of research, the ancient Greeks apparently did not find the dependence of the base tone of the tension of the string (neither they knew the dependence of the density). Probably this is because they did not have any clear concept of force, and no devices to measure it.

4. In the derivation above I followed the book by Gantmakher and Krein, Oscillation Matrices and Kernels and small vibrations of mechanical systems, Translated by AMS from the Russian original of 1941. The authors of this book credit the argument to Daniel Bernoulli (1700-1782).