String with beads

A. Eremenko

August 29, 2024

Sometimes eigenvalues of a large matrix can be found explicitly. Here is an important historical example.

Let us find eigenvalues and eigenvectors of the following $n \times n$ matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

Writing

$$A\mathbf{u} = \lambda \mathbf{u}, \quad \mathbf{u} = (u_1, \dots, u_n)^T,$$

and setting

$$u_0 = 0, \quad u_{n+1} = 0, \tag{1}$$

we obtain a system of equations

$$-u_{j-1} + 2u_j - u_{j+1} = \lambda u_j, \quad 1 \leq j \leq n.$$

This is a two-term recurrence:

$$u_{j+1} + (\lambda - 2)u_j + u_{j-1} = 0,$$

and solutions have to be sought in the form

$$u_j = \rho^j.$$

Plugging this form, we obtain the characteristic equation of the recurrence

$$\rho^2 + (\lambda - 2)\rho + 1 = 0,$$

whose solution is

$$\rho_{1,2} = \frac{2-\lambda}{2} \pm \sqrt{\frac{(2-\lambda)^2}{4} - 1}.$$

If $\rho_{1,2}$ are real, they are of the same sign, since their product is 1, and the general solution of our recurrence will be

$$u_j = c_1\rho_1^j + c_2\rho_2^j, \quad \text{if} \quad \rho_1 \neq \rho_2.$$

trying to satisfy the boundary conditions (1) we obtain $c_1 = c_2 = 0$, so this does not lead to an eigenvalue of the original problem. Check that similar conclusion holds of $\rho_1 = \rho_2$. Therefore, $\rho_{1,2}$ must be non-real. Since we will need their powers, it is useful to express them in the exponentials (polar) form. To do this, set

$$\cos\theta = \frac{2-\lambda}{2}, \tag{2}$$

then

$$\rho_{1,2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}.$$

So the general (real) solution of our two-term recurrence is

$$u_j = c_1\cos j\theta + c_2\sin j\theta. \tag{3}$$

Satisfying the boundary conditions (1) we obtain $c_1 = 0$ and

$$\sin((n+1)\theta) = 0,$$

which gives

$$\theta_k = \frac{\pi k}{n+1}, \quad 1 \leq k \leq n.$$

Using formula (2), and trigonometry, we obtain our eigenvalues:

$$\lambda_k = 2 - 2\cos\theta_k = 4\sin^2\frac{\theta_k}{2}. \tag{4}$$

Notice that eigenvalues are positive, so our matrix is positive definite.

Eigenvectors are given by formula (3) with $c_1 = 0,\ c_2 = 1$:

$$u_{k,j} = \sin j\theta_k = \frac{\pi kj}{n+1}. \tag{5}$$

I recall the mechanical interpretation of our matrix: it describes oscillations of a string with equally spaced beads of equal mass:

$$m\mathbf{y}'' + KA\mathbf{y} = 0, \tag{6}$$

where m is the mass of a bead, and K is the Hook constant.

Let us now try to pass to the limit and obtain oscillations of a continuous string. Suppose that the string is stretched by the force σ, its total length is L, so the length of a piece between the beads is $\ell = L/(n+1)$, and the Hooke constant for a piece of the string is

$$K = \sigma/\ell = \frac{\sigma(n+1)}{L}.$$

Let us introduce the constant linear density ρ, so that the mass of the whole string is ρL, and

$$m = \rho L/n.$$

In these notations, $K/m = \sigma n(n+1)/L^2$, and our equation (6) becomes

$$\mathbf{y}'' + \frac{\sigma n(n+1)}{\rho L^2} A\mathbf{y} = 0.$$

Substituting $\mathbf{y}(t) = e^{i\omega t}\mathbf{u}$, we obtain

$$-\omega^2\mathbf{u} + \frac{\sigma n(n+1)}{\rho L^2} A\mathbf{u} = 0,$$

so

$$\omega_k = \sqrt{\frac{\sigma}{\rho}}\frac{\sqrt{n(n+1)}}{L}\sqrt{\lambda_k},$$

where λ_k is an eigenvalue of A which is given by formula (4). Passing to the limit as $n \to \infty$ we obtain the formula for frequencies of oscillations of the continuous string:

$$\omega_k^* = \sqrt{\frac{\sigma}{\rho}}\frac{\pi k}{L}, \quad k = 1, 2, \ldots.$$

This is called *Mersenne's formula.* It says that frequencies of oscillation of a homogeneous string with clamped ends make an arithmetic progression, whose terms are integer multiples of the *base frequency*

$$\omega_1^* = \sqrt{\frac{\sigma}{\rho}}\frac{\pi}{L}.$$

That the base frequency is inverse proportional to the length was probably the earliest discovery in exact sciences: it is credited to Pythagoras himself. But the ancients could not discover the dependence on the density and tension of the spring, this is due to Marine Mersenne (17th century) who was a music theorist. He found this law empirically. The theoretical derivation presented here is due to Johann Bernoulli.