Entire functions with two radially distributed values

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Abstract

We study entire functions whose zeros and one-points lie on distinct finite systems of rays. General restrictions on these rays are obtained. In particular, we show that the zeros and one-points can lie on two different lines only for quadratic polynomials and exponential functions. Non-trivial examples of entire functions with zeros and one-points on different rays are constructed, using the Stokes phenomenon for second order linear differential equations.

MSC 2010: 30D20, 30D35, 34M40, 30D05.

Keywords: entire function, radially distributed value, linearly distributed value, value distribution, linear differential equation, Stokes phenomenon, spectral determinants.

1 Introduction

The zeros of an entire function can be arbitrarily assigned, but in general one cannot assign the preimages of two values [22]. Since this work of Nevanlinna, various necessary conditions which the sets of zeros and 1-points of an entire function must satisfy were found; see, e.g., [23, 26, 32]. Besides an intrinsic interest, these conditions are relevant to control theory [6, 5, 14].

In this paper we study the simplest setting when the zeros and 1-points lie on finitely many rays, or are close to finitely many rays.

We begin by recalling some classical results. The word "ray" in this paper will always mean a ray from the origin. For an entire function f, we say that

*Supported by NSF grant DMS-1361836.

a value a is *radially distributed* if the set $f^{-1}(a)$ is contained in the union of finitely many rays.

Theorem A. (A. Edrei [12]) *Suppose that all zeros and* 1*-points of an entire function* f *are distributed on a finite set of rays, and let* ω *be the smallest angle between these rays. Then the order of* f *is at most* π/ω*.*

Theorem B. (I. N. Baker [3], T. Kobayashi [19]) *Suppose that all zeros of a transcendental entire function* f *lie on a line* L_1 *and all* 1*-points lie on a different line* L_2 *parallel to* L_1*. Then* $f(z) = P(e^{az})$ *with some* $a \in \mathbb{C}$ *and a polynomial* P*.*

We complement the theorem of Baker and Kobayashi with the following result.

Theorem 1. *Suppose that all zeros of an entire function* f *lie on a line* L_1 *and all* 1*-points lie on a different line* L_2 *intersecting* L_1*. Then* f *is either of the form* $f(z) = e^{az+b}$ *or* $f(z) = 1 - e^{az+b}$*, or a polynomial of degree at most* 2*.*

As a corollary we obtain that if the zeros and 1-points of an entire function f lie on two distinct rays, then f is a polynomial of degree at most 2.

It is remarkable that there are non-trivial examples of entire functions whose zeros lie on the positive ray and all 1-points lie on two rays that are not contained in the real line.

Theorem 2. *For every integer* $m \geq 3$*, there exists an entire function* f *of order* $1/2 + 1/m$ *whose zeros are positive and whose* 1*-points lie on the two rays* $\{z\colon \arg z = \pm 2\pi/(m+2)\}$*.*

Theorem 1 implies that such functions do not exist for $m = 2$. Taking $f(z^n)$ we obtain an entire function whose zeros lie on n rays and whose 1-points lie on $2n$ rays distinct from those rays where the zeros lie.

Now we relax the condition that the zeros and 1-points are radially distributed. We use the standard notation of the theory of entire and meromorphic functions [15]. Let

$$A = \bigcup_{j=1}^{n} A_j, \quad A_j = \{te^{i\alpha_j} : t \geq 0\}, \quad 0 \leq \alpha_1 \ldots < \alpha_n < 2\pi,$$

be a finite union of rays. For $\varepsilon > 0$ let A_ε be the union of the sectors

$$A_\varepsilon = \bigcup_{j=1}^{n} \{z \colon |\arg z - \alpha_j| < \varepsilon\}.$$

We say that the a-points of an entire function f are *close* to the set A if, for every $\varepsilon > 0$, we have

$$n(r, \mathbb{C}\backslash A_\varepsilon, a, f) = o(\log M(r)), \quad r \to \infty,$$

where the left hand side is the number of a-points in $\{z \in \mathbb{C}\backslash A_\varepsilon \colon |z| \leq r\}$ and $M(r) = \max\{|f(z)| \colon |z| \leq r\}$.

Our next result concerns the situation when the zeros are close to a finite union of rays A and the 1-points are close to a finite union of rays B, where the sets A and B are disjoint, apart from the origin. We will assume that the system of rays $A \cup B$ is *minimal* is the sense that for every ray $\{te^{i\alpha} \colon t \geq 0\} \subset A \cup B$ there is a sequence (z_k) tending to ∞ such that $f(z_k) \in \{0, 1\}$ for all k and $\arg z_k \to \alpha$ as $k \to \infty$.

Theorem 3. *Let f be a transcendental entire function of order $\rho < \infty$ whose zeros are close to A and whose 1-points are close to B, with $A \cap B = \{0\}$. Suppose that the system $A \cup B$ is minimal.*

Then

$$\rho = \frac{\pi}{\omega} > \frac{1}{2}, \tag{1}$$

where ω is the largest *angle between adjacent rays in $A \cup B$, and there exists a system of rays $C = \bigcup_{j=1}^{2m} C_j \subset A \cup B$, with $m \geq 1$, partitioning the plane into $2m$ sectors S_j such that $\partial S_j = C_j \cup C_{j+1}$ for $1 \leq j \leq 2m - 1$ and $\partial S_{2m} = C_{2m} \cup C_1$, with the following properties:*

(i) The angle of S_j at 0 is π/ρ when j is even, and at most π/ρ when j is odd.

(ii) Both boundary rays of an odd sector belong to the same set, A or B.

(iii) There are no rays of $A \cup B$ inside the even sectors.

(iv) If there are rays of A inside an odd sector, then the boundary rays of this sector belong to B. If there are rays of B inside an odd sector, then the boundary rays of this sector belong to A.

(v) If there are no rays of $A \cup B$ in an odd sector, then its opening angle is π/ρ.

The next result – whose proof we will only sketch – shows that Theorem 3 is best possible.

Theorem 4. *Let A and B be systems of rays, satisfying conditions (i)–(v) of Theorem 3 for some $\rho \in (0, \infty)$. Then there exists an entire function of order ρ whose zeros are close to A and whose 1-points are close to B.*

Moreover, for all finite systems of rays A and B there exists an entire function of infinite order whose zeros are close to A and whose 1-points are close to B.

We illustrate our results by considering the case of three distinct rays. First we note the following consequence of Theorems 1 and 3.

Corollary. *Let f be a transcendental entire function whose zeros lie on a ray L_0 and whose 1-points lie on two rays L_1 and L_{-1}. Suppose that the numbers of zeros and 1-points are infinite. Then $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$.*

Now we consider three rays

$$L_j = \{te^{ij\alpha} : t \geq 0\}, \quad j \in \{-1, 0, 1\},$$

with $\alpha \in (0, \pi)$. Theorems 1–3 imply the following:

Theorem 2 shows that for certain $\alpha \in (0, \pi/2)$ there exists a transcendental entire function of order $\pi/(2\pi - 2\alpha)$ whose zeros lie on L_0 while its 1-points lie on $L_1 \cup L_{-1}$. It remains open whether this holds for all $\alpha \in (0, \pi/2)$; see the discussion at the end of the paper on possible generalizations of this theorem.

If $\alpha = \pi/2$, then, according to Theorem 1, there is no transcendental entire function with infinitely many zeros on L_0 and infinitely many 1-points on $L_1 \cup L_{-1}$. However, the entire function $f(z) = 1/\Gamma(-z)$ has zeros on L_0 and 1-points close to the imaginary axis. This follows from Stirling's formula.

Finally, Theorem 3 implies that if $\alpha \in (\pi/2, \pi)$, then there is no transcendental entire function whose zeros are close to L_0 and whose 1-points are close to $L_1 \cup L_{-1}$.

Theorems 1 and 2 answer questions 3.1 and 3.2 asked by Gary Gundersen in [16]. We thank him for drawing our attention to these questions and for interesting discussions which stimulated this work.

The plan of the paper is the following. In section 2 we prove Theorem 3 and the corollary. The proof of Theorem 4 showing the sharpness of Theorem 3 is then sketched in section 3. In section 4 we will use Theorem 3 to prove Theorem 1. The proof of Theorem 2 is independent of the rest and will be given in section 5.

2 Proof of Theorem 3 and the corollary

Proof of Theorem 3. If the a-points of f are close to a finite system of rays, then evidently $f(z+c)$ has the same property for every $c \in \mathbb{C}$, with the same rays. Therefore we may assume without loss of generality that

$$f(0) \notin \{0,1\}. \tag{2}$$

Let (r_k) be a sequence tending to ∞ with the property that

$$\log M(tr_k) = O(\log M(r_k)), \quad k \to \infty, \tag{3}$$

for every $t > 0$. Such sequences always exist for functions of finite order.

A sequence (r_k) is called a sequence of *Pólya peaks of order* $\lambda \in [0, \infty)$ for $\log M(r)$, if for every $\varepsilon > 0$ we have

$$\log M(tr_k) \leq (1+\varepsilon) t^{\lambda} \log M(r_k), \quad \varepsilon \leq t \leq \varepsilon^{-1}, \tag{4}$$

when k is large enough. It is clear that every sequence of Pólya peaks satisfies (3). According to a result of Drasin and Shea [11], Pólya peaks of order λ exist for all finite $\lambda \in [\rho_*, \rho^*]$, where

$$\rho^* = \sup \left\{ p > 0\colon \limsup_{r,A \to \infty} \frac{\log M(Ar)}{A^p \log M(r)} = \infty \right\} \tag{5}$$

and

$$\rho_* = \inf \left\{ p > 0\colon \liminf_{r,A \to \infty} \frac{\log M(Ar)}{A^p \log M(r)} = 0 \right\}.$$

We always have

$$0 \leq \rho_* \leq \rho \leq \rho^* \leq \infty,$$

so when $\rho < \infty$, then there exist Pólya peaks of some (finite) order λ.

We refer to [17, Ch. III], [18, Ch. III] and [25] for the basic results on subharmonic functions used below. Fixing a sequence (r_k) with the property (3), we consider the two sequences (u_k) and (v_k) of subharmonic functions given by

$$u_k(z) = \frac{\log |f(r_k z)|}{\log M(r_k)} \quad \text{and} \quad v_k(z) = \frac{\log |f(r_k z) - 1|}{\log M(r_k)}.$$

In view of (3), these sequences are bounded from above on every compact subset of $\mathbb{C}$. It follows from (2) that the sequences $u_k(0)$ and $v_k(0)$ tend to 0. According to a well known compactness principle (see, for example, [17, Theorems 4.1.8, 4.1.9] or [18, Theorems 3.2.12, 3.2.13]), one can choose a subsequence of (r_k), which we do without changing notation, such that the limit functions

$$u(z) = \lim_{k\to\infty} \frac{\log|f(r_k z)|}{\log M(r_k)} \quad \text{and} \quad v(z) = \lim_{k\to\infty} \frac{\log|f(r_k z) - 1|}{\log M(r_k)} \tag{6}$$

exist and are subharmonic. Here the convergence is in the Schwartz space $\mathscr{D}'$. It implies the convergence of the Riesz measures, as the Laplacian is continuous in $\mathscr{D}'$.

The functions u and v are non-zero subharmonic functions, and they have the following properties:

(a) $u^+ = v^+$.

(b) $\{z\colon u(z) < 0\} \cap \{z\colon v(z) < 0\} = \emptyset$.

(c) u is harmonic in $\mathbb{C}\backslash A$ and v is harmonic in $\mathbb{C}\backslash B$.

If (r_k) is a sequence of Pólya peaks of order $\lambda > 0$, then we have the additional property

(d) $u(0) = v(0) = 0$, and $\max\{u(z), v(z)\} \le |z|^\lambda$ for all $z \in \mathbb{C}$.

Properties (a) and (b) are evident. Property (c) holds because the Laplacian is continuous in $\mathscr{D}'$. Property (d) is a consequence of (2) and (4). Indeed, (2) and

$$u(0) \ge \limsup_{k\to\infty} u_k(0),$$

(see [17, (4.1.8)]) imply that $u(0) \ge 0$, while (4) yields $u(z) \le |z|^\lambda$ and thus, in particular, $u(0) = 0$. The same argument applies to v.

The components of the complement $\mathbb{C}\backslash(A \cup B)$ will be called *sectors of the system* $A \cup B$.

Lemma 1. *Let u and v be two non-zero subharmonic functions in the plane which satisfy (a), (b) and (c). Then either $u(z) \equiv v(z) \equiv c$ for some $c > 0$, or there exist an even number of rays $C_1, \dots, C_{2m}$, with $m \ge 1$, that belong to $A \cup B$ and partition the plane into sectors S_j, so that $\partial S_j = C_j \cup C_{j+1}$ for $1 \le j \le 2m-1$ and $\partial S_{2m} = C_{2m} \cup C_1$, such that $u(z) = v(z) > 0$ for z in the even sectors while $u(z) \le 0$ and $v(z) \le 0$ for z in the odd sectors.*

If u and v are given by (6)*, where f is the function from Theorem* 3 *and* (r_k) *is a sequence satisfying* (3)*, then in each odd sector, one of the functions u and v is negative while the other one is equal to zero. Moreover, properties* (ii) *and* (iv) *of Theorem* 3 *hold.*

Proof of Lemma 1. If D is a sector of the system $A \cup B$, and if at some point z_0 in D, we have $\max\{u(z_0), v(z_0)\} > 0$, then $u(z) = v(z) > 0$ for all points $z \in D$. Indeed, both u and v are harmonic in D by (c), and (a) gives

$$u(z_0) = v(z_0) > 0. \tag{7}$$

If $\min\{u(z_1), v(z_1)\} < 0$ for some $z_1 \in D$, then this also holds in a neighborhood of z_1, and one of the functions u and v must be zero in this neighborhood by (b). Then it is identically equal to zero in D which contradicts (7). Thus u and v are non-negative in D, and the minimum principle implies that they are positive. Then they are equal in D by (a). Such sectors D will be called *positive* sectors.

If one of the functions u and v is constant, then both functions are equal to the same positive constant. This follows from (a) and (b). For the rest of the proof we assume that they are non-constant.

Suppose that some ray $L \subset A \cup B$ has the property that positive sectors D_1 and D_2 are adjacent to L on both sides, that is, $L = \partial D_1 \cap \partial D_2$. (We will see in a moment that $D_1 \neq D_2$). Then we have $u(z) = v(z)$ for $z \in D = D_1 \cup D_2 \cup L$, in view of (a), and u and v must be positive and harmonic in D in view of (c).

If there are no non-positive sectors, then u and v are equal, positive and harmonic is $\mathbb{C}\backslash\{0\}$, which is impossible under the current assumption that they are non-constant. So there is at least one non-positive sector. In particular, $D_1 \neq D_2$ in the previous paragraph.

Let D be a positive sector. Let z_0 be a point on $\partial D \cap \partial D'$, where D' is a non-positive sector. This means that $u(z) \leq 0$ and $v(z) \leq 0$ in D'. Then $u(z_0) = v(z_0) = 0$. Indeed, $u(z_0) = v(z_0) \geq 0$ by the upper semi-continuity of subharmonic functions. As D' is not thin at z_0 (in the sense of potential theory, see [25]) we obtain that $u(z_0) = v(z_0) = 0$.

Let C be the union of those rays in $A \cup B$ which separate a positive and a non-positive sector. It follows from the above considerations that C can be written in the form $C = \bigcup_{j=1}^{2m} C_j$, with $m \geq 1$, with rays $C_j \subset A \cup B$ so that in the sector S_j between C_j and C_{j+1} the functions u and v are positive for even j and non-positive for odd j. Moreover, we have $u(z) = v(z) = 0$ for

$z \in C$. Note that the sectors with respect to the system C may be unions of several sectors and rays of the system $A \cup B$.

Suppose now that u and v are given by (6), where the sequence (r_k) satisfies (3). Let S_{2j-1} be an odd sector. Then $u(z) \leq 0$ in S_{2j-1}. If $u(z) = 0$ in S_{2j-1}, then u is not harmonic on either of the two rays in ∂S_{2j-1}, so f has infinitely many zeros close to these two rays. Therefore f cannot have infinitely many 1-points close to either one of these two rays, and thus v is harmonic in a neighborhood of $\partial S_{2j-1} \backslash \{0\}$. As $v(z) = 0$ on ∂S_{2j-1}, we conclude that $v(z) < 0$ in S_{2j-1}. The same argument applies with the roles of u and v interchanged. Thus in each odd sector S_{2j-1} one of the two functions u and v is strictly negative, and the other function is equal to zero. If $u(z) < 0$ in S_{2j-1} then both rays of ∂S_{2j-1} belong to B and all rays of $A \cup B$, if any, inside S_{2j-1} belong to A, and analogously if $v(z) < 0$ in S_{2j-1}. This proves Lemma 1.

We return to the proof of Theorem 3. Lemma 1 does not exclude the possibility that the set of rays C_j is empty, and thus the whole plane coincides with one positive sector. In this case u and v are identically equal to the same positive constant. The following argument shows that this is impossible.

We will in fact show that there are no rays of $A \cup B$ inside the even sectors S_{2j}, that is, the even sectors of the system C coincide with the positive sectors of the system $A \cup B$. Consider an even sector S_{2j}. As u is positive and harmonic in S_{2j} and zero on the boundary, it must have the form

$$u(re^{it}) = c_j r^{\gamma_j} \cos(\gamma_j t - t_j), \tag{8}$$

where π/γ_j is the angle of this sector at the origin. This can be seen by transforming the sector S_{2j} to a half-plane, for which the result is standard [7, Theorem I]. For a given system $A \cup B$, there are only finitely many possibilities for these numbers γ_j. Thus if (r_k) is a sequence of Pólya peaks of order $\lambda > 0$, so that (d) holds, we obtain by comparing (d) with (8) that $\gamma_j = \lambda$ for all j. As the possible values of λ always fill a closed set $[\rho_*, \rho^*]$, where $\rho^* \leq +\infty$, we conclude that this closed set is degenerate to a point, that is, $\rho^* = \rho_* = \rho$. In particular, $\rho^* < \infty$, and (5) implies that *every* sequence (r_k) tending to ∞ satisfies (4). Also this shows that the angle of every even sector at the origin is equal to π/ρ, proving the first statement of (i).

For $r_0 > 0$ such that $M(r_0) > 1$ we consider the curve mapping $[r_0, \infty)$

to $\mathscr{D}' \times \mathscr{D}'$ given by

$$r \mapsto \left(\frac{\log |f(rz)|}{\log M(r)}, \frac{\log |f(rz) - 1|}{\log M(r)} \right).$$

Let F be the limit set of this curve when $r \to \infty$. It consists of pairs (u, v) satisfying (a), (b) and (c) and thus satisfying the conclusions of Lemma 1. As a limit set of a curve, F is connected. In each sector of the system $A \cup B$ either both of the functions u and v are positive, or one is negative. We conclude that the sectors S_j can be chosen independently of the sequence (r_k).

Now suppose that a ray $L = \{te^{i\beta} \colon t \geq 0\}$ of the set A lies inside an even sector S_{2j}. By assumption, there is an infinite sequence of zeros (z_k) of the form $z_k = r_k e^{i\beta_k}$ with $r_k \to \infty$ and $\beta_k \to \beta$. Passing to a subsequence we may assume that the limits in (6) exist. Let

$$u_k(z) = \frac{\log |f(r_k z)|}{\log M(r_k)}.$$

We have $u_k \to u$ in $\mathscr{D}'$. According to Azarin [1], this convergence also holds in the following sense: for every $\varepsilon > 0$ the set

$$\{z \colon |u(z) - u_k(z)| > \varepsilon\}$$

can be covered by discs the sum of whose radii is at most ε, when k is large enough. Let D be the closed disk with the center at $e^{i\beta}$ of radius δ so small that $D \subset S_{2j}$. Then $\mu := \min\{u(z) \colon z \in D\} > 0$. Choosing $\varepsilon < \min\{\delta/2, \mu/2\}$ we see that, for each large k, there is a circle T_k around $e^{i\beta}$ such that $z_k/r_k = e^{i\beta_k} \in B_k \subset D$, where B_k is the disk with $T_k = \partial B_k$, and

$$u_k(z) \geq \mu/2, \quad z \in T_k.$$

This means that $\log |f(z)| \geq (\mu/2) \log M(r_k)$ for z on the circles

$$\{z \colon z/r_k \in T_k\}. \tag{9}$$

Each of these circles encloses a zero of f, namely z_k. Thus, by Rouche's theorem, each of them also contains a 1-point. This is a contradiction, because the circles (9) remain inside closed subsectors of S_{2j} that do not contain rays from B. A similar argument shows that there are no rays from B inside any even sector S_{2j}.

Thus no rays of the system $A \cup B$ are contained in the even sectors. As f is transcendental, Picard's theorem yields that the system $A \cup B$ contains at least one ray. We conclude that u and v are not constant, no matter what sequence (r_k) was used to define them. This also implies (1), and proves (iii) and the fact that the even sectors of the system C coincide with the positive sectors of the system $A \cup B$.

It remains to prove (v) and the second statement of (i). Let S_{2j-1} be an odd sector with angle π/γ at the origin. Let us consider again the limit functions u and v obtained from the Pólya peaks r_k. Then we have (d) with $\lambda = \rho$. One of the two subharmonic functions, say u, is negative in S_{2j-1}, and zero on the boundary of S_{2j-1}. Let h be the least harmonic majorant of u in $G = S_{2j-1} \cap \{z \colon |z| < 1\}$. Then h is a negative harmonic function in G, equal to zero on the straight segments of ∂G. Similarly as in (8) it follows that

$$\int_{re^{it} \in G} h(re^{it})\, dt \leq -cr^{\gamma}, \quad r < 1,$$

where $c > 0$. Then u satisfies the same inequality, and combined with property (d) and using

$$0 = u(0) \leq \int_0^{2\pi} u(re^{it})\, dt$$

this implies that $\gamma \geq \rho$ so that $\pi/\gamma \leq \pi/\rho$. This proves the second part of (i).

Finally, if there are no rays of A inside S_{2j-1}, then u is harmonic in S_{2j-1}, so it is of the form (8), and it is a harmonic continuation from an adjacent even sector, so we must have $\gamma = \rho$. A similar argument applies if v is negative and there are no rays of B inside S_{2j-1}. This proves (v) and completes the proof of Theorem 3.

Proof of the Corollary. We assume without loss of generality that L_0 is the positive ray. The order of f must be finite by Theorem A, so Theorem 3 is applicable. As there are only three rays, the number m in Theorem 3 must be 1. So we have one even sector of opening π/ρ and one odd sector of opening at most π/ρ. In view of (ii), the common boundary of the odd and even sector is $L_1 \cup L_{-1}$. So L_0 lies inside the odd sector. Thus $\rho \leq 1$. The possibility that $\rho = 1$ is excluded by (1) and Theorem 1. As $\rho < 1$, the function f is of genus zero and thus of the form

$$f(z) = cg(z), \quad g(z) = z^n \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_k}\right),$$

where n is a non-negative integer and (z_k) is a sequence of positive numbers tending to ∞. If c is real, we conclude that the rays L_1 and L_{-1} are symmetric with respect to L_0 which proves the corollary in this case.

Suppose now that L_1 and L_{-1} are not symmetric with respect to L_0 so that c is not real. Let us set $a = 1/c$. Then $f(z) = 1$ is equivalent to $g(z) = a$. We consider the function $h(z) = (g(z) - a)/(\overline{a} - a)$. In view of the symmetry of g, the zeros of h lie on the rays L_1 and L_{-1}, while the 1-points lie on the reflected rays $\overline{L_1}$ and $\overline{L_{-1}}$. Since L_0 lies in the odd sector, which has angle $< \pi$ at the origin, it follows that the two rays L_1, L_{-1} are interlaced with the two rays $\overline{L_1}$, $\overline{L_{-1}}$. This contradicts (*ii*) and (*iii*) of Theorem 3, and completes the proof of the corollary.

3 Sketch of the proof of Theorem 4

We only indicate the construction of examples showing that Theorem 3 is best possible, as this construction is well-known, see for example [10], where a similar construction was used for the first time.

We fix $\rho \in (1/2, \infty)$ and construct a ρ-trigonometrically convex function h such that the union of the even sectors coincides with the set

$$\{re^{it} \colon r > 0,\ h(t) > 0\},$$

and such that h is trigonometric except at the arguments of some rays inside the odd sectors. If there are no rays of $A \cup B$ in the odd sectors at all, then ρ must be an integer, and we just take h to be of the form (8) with $\gamma_j = \rho$.

Then we discretize the Riesz mass of the subharmonic function

$$w(re^{it}) = r^{\rho} h(t),$$

as it is done in [1], and obtain an entire function g with zeros on some rays $A \cup B$ which lie in the odd sectors, and such that

$$\lim_{r \to \infty} r^{-\rho} \log |f(rz)| = w(z).$$

If there are odd sectors with opening π/ρ, then h must be trigonometric on the intervals corresponding to these sectors, so we multiply f by a canonical product of order smaller than ρ to achieve that f has infinitely many zeros on all those rays of the system $A \cup B$ which belong to the odd sectors. Then we label the odd sectors with labels 0 and 1: if the boundary of an odd sector belongs to A, we label it with 1, and if the boundary belongs to B we label it with 0.

Let S_j be an odd sector labeled with 1. Consider the component D_j of the set $\{z\colon |f(z)| < 2\}$ which is asymptotic to S_j. Let p be a quasiconformal map of the disk $\{z\colon |z| < 2\}$ onto itself, equal to the identity mapping on the boundary, and such that $p(0) = 1$, whose complex dilatation is supported by the set $\{z\colon 3/2 \leq |z| \leq 2\}$. We define

$$G(z) = \begin{cases} p(g(z)), & z \in \bigcup_j D_j, \\ g(z), & \text{otherwise.} \end{cases}$$

Here the union is over all odd sectors labeled with 1. This G is a quasiregular map of the plane, whose dilatation is supported by a small set E in the sense that

$$\int_E \frac{dxdy}{x^2 + y^2} < \infty.$$

Then the theorem of Teichmüller–Wittich–Belinski [20, §V.6] guarantees the existence of a quasiconformal map ϕ such that $f = G \circ \phi$ is an entire function, and $\phi(z) \sim z$ as $z \to \infty$. It is easy to verify that f has all the required properties.

For the construction of infinite order functions, let $A = \bigcup_{j=1}^m \{te^{i\alpha_j}\colon t \geq 0\}$ and $B = \bigcup_{k=1}^n \{te^{i\beta_k}\colon t \geq 0\}$ be two finite systems of rays with $A \cap B = \{0\}$. Again we only sketch the argument.

First we note that by [24, Part III, Problems 158–160] there exists an entire function E such that $z^2(E(z) + 1/z)$ is bounded outside the half-strip $S = \{z\colon \operatorname{Re} z > 0, |\operatorname{Im} z| < \pi\}$. In particular, E is bounded outside S. Considering $F(z) = \delta(E(z) - c)/((z - a)(z - b))$, where $\delta > 0$ is small, $c \in \mathbb{C}$, and a and b are c-points of E, we obtain an entire function F such that

$$|F(z)| \leq \frac{1}{|z|^2} \leq \frac{1}{\operatorname{dist}(z, S)^2}, \quad z \notin S,$$

where $\operatorname{dist}(z, S)$ denotes the distance from z to S. For some large $R > 0$ we now consider the functions

$$a_j(z) = 1 + z \exp F(e^{-i\alpha_j} z - R) \quad \text{and} \quad b_k(z) = z \exp F(e^{-i\beta_k} z - R).$$

With $S_j = \{e^{i\alpha_j}(z+R)\colon z \in S\}$ we find, noting that $|e^w - 1| \leq 2|w|$ for $|w| \leq 1$, that

$$\begin{aligned} |a_j(z) - z - 1| &\leq \left| z \left(\exp F(e^{-i\alpha_j} z - R) - 1 \right) \right| \leq 2|zF(e^{-i\alpha_j} z - R)| \\ &\leq \frac{2|z|}{\operatorname{dist}(e^{-i\alpha_j} z - R, S)^2} = \frac{2|z|}{\operatorname{dist}(z, S_j)^2}, \quad z \notin S_j. \end{aligned}$$

Similarly, with $T_k = \{e^{i\beta_k}(z+R)\colon z \in S\}$ we have

$$|b_k(z) - z| \leq \frac{2|z|}{\operatorname{dist}(z, T_k)^2}, \quad z \notin T_k.$$

We choose $\varepsilon > 0$ so small that the sectors $U_j = \{z\colon |\arg(z - e^{i\alpha_j} R/2) - \alpha_j| \leq \varepsilon\}$ and $V_k = \{z\colon |\arg(z - e^{i\beta_k} R/2) - \beta_k| \leq \varepsilon\}$ are disjoint and put

$$G(z) = \begin{cases} a_j(z), & z \in U_j, \\ b_k(z), & z \in V_k. \end{cases}$$

Then

$$G(z) = z + O(1), \quad z \in \bigcup_{j=1}^{m} \partial U_j \cup \bigcup_{k=1}^{n} \partial V_k.$$

This allows to extend G to a quasiregular map of the plane which satisfies

$$G(z) = z + O(1), \quad z \in \mathbb{C} \backslash \left(\bigcup_{j=1}^{m} U_j \cup \bigcup_{k=1}^{n} V_k \right)$$

and whose dilatation K_G satisfies $K_G(z) = 1 + O(1/|z|)$ as $z \to \infty$. Again the theorem of Teichmüller–Wittich–Belinski yields the existence of a quasiconformal map ϕ such that $f = G \circ \phi$ is entire and $\phi(z) \sim z$ as $z \to \infty$. It is not difficult to show that the zeros of f are close to A and the 1-points of f are close to B.

We note that the method does not actually require that the rays that form A are distinct from those that form B. Indeed, if we want that both zeros and 1-points accumulate at $\{te^{i\alpha_j}\colon t \geq 0\}$, we only have to choose $a_j(z) = c + z \exp F(e^{-i\alpha_j} z - R)$ with a constant c different from 0 and 1.

4 Proof of Theorem 1

According to Theorem A, the order ρ of f is finite.

First we deal with the case when f is a polynomial, following Baker [2]. Without loss of generality, we may assume that L_1 is the real line. Then $f = cg$, where g is a real polynomial with all zeros real. Then all zeros of f' are real. Similarly we conclude that all zeros of f' lie on L_2, and hence the point z_0 of intersection of L_1 and L_2 is the only possible zero of f'. Hence $f(z) = c_1(z - z_0)^n + c_2$ for some $n \geq 1$ and some $c_1, c_2 \in \mathbb{C}$ with $c_1 \neq 0$. Such a function f can satisfy the assumptions of Theorem 1 only if f is a polynomial of degree at most 2. Notice that this argument can be extended to functions of order less than 2, but we do not use this.

Suppose now that f is transcendental. Then we use Theorem 3. This theorem implies that there exists at least one even sector. If there is only one even sector, and its angle is greater than π, then the odd sector does not contain rays of $A \cup B$, so by (v) its opening must be the same as the opening of the even sector, which is a contradiction.

If there are two even sectors, then the odd sectors contain no rays of $A \cup B$. It follows from (i) and (v) that all sectors must have opening $\pi/2$. Then by (ii) the zeros are close to the boundary of a quadrant, and the 1-points are close to the the boundary of the opposite quadrant. But by hypothesis the zeros lie on a line and the 1-points lie on a line. We conclude that the zeros are actually close to one ray and the 1-points are close to another ray. But then there is only one even sector, contradicting our assumption at the beginning of this paragraph.

The only remaining possibility is that there is one even sector with opening π. Then $\rho = 1$, and we assume without loss of generality that this sector is the upper half plane and the 1-points are real. This means that B consists of two rays whose union is the real line. Using the notation of the proof of Theorem 3, and choosing a sequence (r_k) of Pólya peaks of order 1, we obtain $u(z) = \operatorname{Im} z$ and $v(z) = \operatorname{Im}^+ z$. This implies that

$$N(r_k, 1, f) \sim \frac{1}{\pi} \log M(r_k), \quad k \to \infty. \tag{10}$$

Now let $g(z) = f(z)\overline{f(\overline{z})}$. As all 1-points of f are real, $f(z) = 1$ implies that $g(z) = 1$, so if $g \not\equiv 1$, we will have from (10) that

$$N(r_k, 1, g) \geq (1 - o(1)\frac{1}{\pi} \log M(r_k), \quad k \to \infty. \tag{11}$$

Now define the subharmonic function

$$w(z) = \lim_{k\to\infty} \frac{\log|g(r_k z)|}{\log M(r_k)}.$$

It is evident that $w(z) = u(z) + u(\overline{z}) = 0$. Together with (11) this implies that $g(z) \equiv 1$. We conclude that with this normalization, f has the form $f(z) = \exp(icz + id)$, where c and d are real. This completes the proof of Theorem 1.

5 Proof of Theorem 2

We consider differential equations

$$-y'' + \left((-1)^\ell z^m + E\right) y = 0, \quad \ell \in \{0, 1\}, \quad m \geq 3, \quad E \in \mathbb{C}. \tag{12}$$

Here m is an integer, so all solutions are entire functions. The equation has the following symmetry property. Set

$$\varepsilon = e^{\pi i/(m+2)}, \quad \omega = \varepsilon^2.$$

If $y_0(z, E)$ is a solution of (12) then

$$y_k(z, E) = y_0(\omega^{-k} z, \omega^{2k} E) \tag{13}$$

satisfies the same equation, while

$$y_0(\varepsilon^{-k} z, \varepsilon^{2k} E)$$

with an odd k satisfies (12) with the sign at z^m switched.

The Stokes sectors are defined as follows. When $\ell = 0$, they are $S_0 = \{z\colon |\arg z| < \pi/(m+2)\}$, and $S_k = \omega^k S_0$ for $k \in \mathbb{Z}$. When $\ell = 1$, the Stokes sectors are $S_0 = \{z\colon 0 < |\arg z| < 2\pi/(m+2)\}$ and $S_k = \omega^k S_0$ for $k \in \mathbb{Z}$.

To obtain a discrete sequence of eigenvalues, one imposes boundary conditions of the form

$$y(z) \to 0, \quad z \to \infty, \quad z \in S_n \cup S_k, \tag{14}$$

for some n and k. The exact meaning of (14) is that $y(z) \to 0$ when $z \to \infty$ along any interior ray from the origin contained in the union of the two sectors.

We will denote such a boundary condition by (n, k). It is known [28] that when $n \neq k \pm 1$ (modulo $m+2$), then the boundary value problem (n, k) has a discrete spectrum with a sequence of eigenvalues tending to infinity. (For completeness, we include the argument below.) Moreover, K. Shin [27] proved that these eigenvalues always lie on a ray from the origin. In particular, when S_n and S_k are symmetric with respect to the positive ray, these eigenvalues are positive. All other cases can be reduced to this case using the symmetry of the differential equation stated above: if ω_1 and ω_2 are bisectors of S_m and S_n, then the eigenvalues lie on the ray $\{t/(\omega_1\omega_2) \colon t \geq 0\}$.

From now on we assume that $\ell = 0$ in (12). For each E the equation (12) has a solution tending to zero as $z \to \infty$ in S_0. More precisely, there is a unique solution $y_0(z, E)$ satisfying

$$y_0(z, E) = (1 + o(1)) z^{-m/4} \exp\left(-\frac{2}{m+2} z^{(m+2)/2}\right) \tag{15}$$

as $z \to \infty$ in any closed subsector of $S_0 \cup S_1 \cup S_{-1}$; see [28, Thm 6.1]. Notice the simple but important fact that this principal part of the asymptotics does not depend on E. The function $y_0(z, E)$ is actually an entire function of the two variables z and E, and its asymptotics when $E \to \infty$ while z is fixed are also known [28, Thm 19.1]; this implies that the entire function $E \mapsto y_0(z_0, E)$ has order

$$\rho = \frac{1}{2} + \frac{1}{m}.$$

Now we define y_k by (13). Then $y_k \to 0$ as $z \to \infty$ in S_k. The boundary problem (n, k) thus has a solution when y_n and y_k are linearly dependent as functions of z. This means that their Wronskian vanishes. But the Wronskian, evaluated at $z = 0$, is an entire function of E, and its order is less than 1. Thus its zeros, which are the eigenvalues of the problem, form a sequence tending to infinity, as mentioned above.

As y_0, y_1, y_{-1} satisfy the same differential equation, we have

$$y_{-1} = C(E) y_0 + \tilde{C}(E) y_1.$$

The asymptotics of y_1 and y_{-1} in S_0 (which follow from (15)) show that $\tilde{C} = -\omega$, so

$$y_{-1} = C(E) y_0 - \omega y_1. \tag{16}$$

By differentiating this with respect to z we obtain

$$y_{-1}' = C(E) y_0' - \omega y_1'. \tag{17}$$

Solving (16) and (17) by Cramer's rule, we obtain

$$C(E) = W_{-1,1}/W_{0,1},$$

where $W_{i,j}$ is the Wronskian of y_i and y_j. This shows that C is an entire function (because $W_{0,1}$ is never 0). It has the same order ρ that y_0 has as a function of E.

In view of (16), the zeros of C are exactly the eigenvalues λ_j of the problem (14) with $(n,k) = (-1,1)$. So all zeros of C are positive by Shin's result. Substituting $(z,E) \mapsto (\omega^{-1}z, \omega^2 E)$ to (16), we obtain

$$y_0 = C(\omega^2 E)y_1 - \omega y_2.$$

Using this to eliminate y_0 from (16) we obtain

$$y_{-1} = \left(C(E)C(\omega^2 E) - \omega\right) y_1 - C(E)\omega y_2.$$

We conclude that the zeros of the entire function

$$g(E) := C(E)C(\omega^2 E) - \omega \tag{18}$$

are the eigenvalues of the problem $(-1,2)$. Therefore, these zeros lie on the ray $\{z = t\omega^{-1} \colon t \geq 0\}$. So if we define $f(E) = -\omega^{-1}g(\omega^{-1}E)$ and $h(E) = C(E)/\sqrt{\omega}$, then

$$f(E) = 1 - h(\omega^{-1}E)h(\omega E),$$

the zeros of f are on the positive ray and the 1-points on two other rays. This completes the proof of Theorem 2.

Remarks. Once it is known that two entire functions C and g satisfy (18) and zeros of each function lie on a ray, the order of both functions and the angles between the rays can be determined from Theorem 3.

Equations of the type (18) occur for the first time in the work of Sibuya and his students [28, 30, 29] for the simplest case when $m = 3$.

It was later discovered that these equations also arise in the context of exactly solvable models of statistical mechanics on two-dimensional lattices and in quantum field theory [8, 9].

The interesting question is to which angles Theorem 2 generalizes. If $m > 2$ is not an integer, equation (12) and its solutions are defined on the Riemann surface of the logarithm, but Sibuya's solution y_0 is still entire as a function of E. We found no source where this fact is proved, but it is stated and used in [8, p. 576], [9, p. R231] and [31]. Shin's result, which we used above seems to generalize to non-integer $m \geq 4$, see [27, Theorem 11] which we use with $\ell = 1$ and $\ell = 2$. On the other hand, numerical evidence in [4] (see Figs. 14, 15, 20) shows that for $m < 4$ our function $g(E)$ in (18) does not have radially distributed zeros on one ray, even if finitely many of the zeros are discarded.

And of course, it would be interesting to know whether there are any other entire functions like in Theorem 2, not related to the differential equations (12).

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