Real entire functions with real zeros and a conjecture of Wiman

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1 Introduction

An entire function is called real if it maps the real line into itself. The main result of this paper is

Theorem 1.1 *For every real entire function of infinite order with only real zeros, the second derivative has infinitely many non-real zeros.*

This conclusion is not true for the first derivative as the example $\exp(\sin z)$ shows. For real entire functions with finitely many zeros, all of them real, Theorem 1.1 was proved in [3]. Theorem 1.1 can be considered as an extension to functions of infinite order of the following result of Sheil-Small [20], conjectured by Wiman in 1914 [1, 2]. For every integer $p \geq 0$, denote by V_{2p} the set of entire functions of the form

$$f(z) = \exp(-az^{2p+2})g(z),$$

where $a \geq 0$ and g is a real entire function with only real zeros of genus at most $2p+1$, and set $U_0 = V_0$ and $U_{2p} = V_{2p} \backslash V_{2p-2}$ for $p \geq 1$. Thus the class of all real entire functions of finite order with real zeros is represented as a union of disjoint subclasses $U_{2p},\ p = 0, 1, \ldots.$

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Theorem A (Sheil-Small) *If $f \in U_{2p}$ then f'' has at least $2p$ non-real zeros.*

Applying Theorem 1.1 to functions of the form

$$f(z) = \exp \int_0^z g(\zeta)\, d\zeta$$

we obtain

Corollary 1.1 *For every real transcendental entire function g, the function $g' + g^2$ has infinitely many non-real zeros.*

For polynomials g the corresponding result was conjectured in [4, Probl. 2.64 and 4.28] and proved in [20]: *If g is a real polynomial then $g' + g^2$ has at least* $\deg g - 1$ *non-real zeros.* Corollary 1.1 also follows from the result of Bergweiler and Fuchs [3].

Theorems 1 and A together imply the following

Corollary 1.2 *If f is a real entire function and ff'' has only real zeros then $f \in U_0$.*

We recall that U_0, the *Laguerre–Pólya class*, coincides with the closure of the set of all real polynomials with only real zeros, with respect to uniform convergence on compact subsets of the plane. This was proved by Laguerre [12] for the case of polynomials with positive zeros and by Pólya [18] in the general case. It follows that U_0 is closed under differentiation, so that all derivatives of a function $f \in U_0$ have only real zeros. Pólya [18] asked if the converse is true: *if all derivatives of a real entire function f have only real zeros then $f \in U_0$.* This conjecture was proved by Hellerstein and Williamson [8, 9]. More precisely, they showed that *for a real entire function f, the condition that $ff'f''$ has only real zeros implies $f \in U_0$.* Our Corollary 1.2 shows that in this result one can drop the assumption on the zeros of f', as Hellerstein and Williamson conjectured [4, Probl. 2.64].

For the early history of results on the conjectures of Wiman and Pólya we refer to [8, 15], which contain ample bibliography. The main result of Levin and Ostrovskii [15] is

Theorem B *If f is a real entire function and all zeros of ff'' are real then*

$$\log^+ \log^+ |f(z)| = O(|z| \log |z|), \quad z \to \infty. \tag{1}$$

This shows that a function satisfying the assumptions of Corollary 1.2 cannot grow too fast, but there is a gap between Theorem B and Theorem A. Our Theorem 1.1 bridges this gap.

One important tool brought by Levin and Ostrovskii to the subject was a factorization of the logarithmic derivative of a real entire function f with only real zeros:

$$\frac{f'}{f} = \psi\phi,$$

where ϕ is a real entire function, and either ψ is a meromorphic function which maps the upper half-plane $H = \{z : \operatorname{Im} z > 0\}$ into itself or $\psi \equiv 1$. This factorization was used in all subsequent work in the subject. A standard estimate for analytic functions mapping the upper half-plane into itself shows that ψ is neither too large nor too small away from the real axis, so the asymptotic behavior of f'/f mostly depends on that of ϕ. One can show that f is of finite order if and only if ϕ is a polynomial.

The second major contribution of Levin and Ostrovskii was the application of ideas from the value distribution theory of meromorphic functions [5, 7, 16]. Using Nevanlinna theory, Hayman [6] proved that for an entire function f, the condition $f(z)f''(z) \neq 0,\ z \in \mathbb{C}$, implies that f'/f is constant. The assumptions of Theorem B mean that $f(z)f''(z) \neq 0$ in H. Levin and Ostrovskii adapted Hayman's argument to functions in a half-plane to produce an estimate for the logarithmic derivative. An integration of this estimate gives (1). To estimate the logarithmic derivative using Hayman's argument they applied an analogue of the Nevanlinna characteristic for meromorphic functions in a half-plane, and proved an analogue of the main technical result of Nevanlinna theory, the lemma on the logarithmic derivative. This characteristic has two independent origins, [13] and [21], and the name "Tsuji characteristic" was introduced in [15].

In this paper we use both main ingredients of the work of Levin and Ostrovskii, the factorization of f'/f and the Tsuji characteristic.

Another important tool comes from Sheil-Small's proof of Theorem A. His key idea was the study of topological properties of the auxiliary function

$$F(z) = z - \frac{f(z)}{f'(z)}.$$

In the last section of his paper, Sheil-Small discusses the possibility of extension of his method to functions of infinite order, and proves the fact which turns out to be crucial: *if f is a real entire function, ff'' has only real zeros, and f' has a non-real zero, then F has a non-real asymptotic value.* In §4 we prove a generalization of this fact needed in our argument.

The auxiliary function F appears when one solves the equation $f(z) = 0$ by Newton's method. This suggests the idea of iterating F and using the Fatou–Julia theory of iteration of meromorphic functions. This was explored by Eremenko and Hinkkanen, see, for example, [10].

Theorem 1.1 will be proved by establishing a more general result conjectured by Sheil-Small [20]. Let L be a real meromorphic function in the plane with only simple poles, all of them real and with positive residues. It is known [8, 15, 20] that every such L has a Levin–Ostrovskii representation

$$L = \psi\phi \tag{2}$$

in which:

(a) ψ is meromorphic in the plane and real on the real axis;
(b) ψ maps the upper half-plane into itself, or $\psi \equiv 1$;
(c) ψ has a simple pole at every pole of L, and no other poles;
(d) ϕ is a real entire function.

We outline briefly how such a factorization (2) is obtained. Let

$$\ldots < a_{k-1} < a_k < a_{k+1} < \ldots$$

be the sequence of poles of L enumerated in increasing order. The assumption that all poles are simple and have positive residues implies that there is at least one zero of L in each interval (a_k, a_{k+1}). We choose one such zero in each interval and denote it by b_k. Then we set

$$\psi(z) = \prod_k \frac{1 - z/b_k}{1 - z/a_k},$$

with slight modifications if $a_k b_k \leq 0$ for some k or the set $\{a_k\}$ is bounded above. We then define ϕ by (2), and properties (a)–(d) follow (for the details see [8, 15, 20]).

Theorem 1.2 *Let L be a function meromorphic in the plane, real on the real axis, such that all poles of L are real, simple and have positive residues. Let ψ, ϕ be as in* (2) *and* (a), (b), (c), (d)*. If ϕ is transcendental then $L + L'/L$ has infinitely many non-real zeros.*

To deduce Theorem 1.1 from Theorem 1.2 it suffices to note that if f is a real entire function with only real zeros then $L = f'/f$ is real meromorphic with only real simple poles and positive residues, and thus has a representation (2). Further, $L + L'/L = f''/f'$. By an argument of Hellerstein and Williamson [9, pp. 500-501], the function ϕ is transcendental if and only if f has infinite order.

2 Preliminaries

We will require the following well known consequence of Carleman's estimate for harmonic measure.

Lemma 2.1 *Let u be a non-constant continuous subharmonic function in the plane. For $r > 0$ let $B(r, u) = \max\{u(z) : |z| = r\}$, and let $\theta(r)$ be the angular measure of that subset of the circle $C(0, r) = \{z \in \mathbb{C} : |z| = r\}$ on which $u(z) > 0$. Define $\theta^*(r)$ by $\theta^*(r) = \theta(r)$, except that $\theta^*(r) = \infty$ if $u(z) > 0$ on the whole circle $C(0, r)$. Then if $r > 2r_0$ and $B(r_0, u) > 1$ we have*

$$\log \|u^+(4re^{i\theta})\| \geq \log B(2r, u) - c_1 \geq \int_{2r_0}^{r} \frac{\pi dt}{t\theta^*(t)} - c_2,$$

in which c_1 and c_2 are absolute constants, and

$$\|u^+(re^{i\theta})\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \max\{u(re^{i\theta}),\, 0\}\, d\theta.$$

The first inequality follows from Poisson's formula, and for the second we refer to [22, Thm III.68]. Note that in the case that $u = \log |f|$ where f is an entire function, $\|u^+(re^{i\theta})\|$ coincides with the Nevanlinna characteristic $T(r, f)$.

Next, we need the characteristic function in a half-plane as developed by Tsuji [21] and Levin and Ostrovskii [15] (see also [5] for a comprehensive treatment). Let f be a meromorphic function in a domain containing the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$. For $t \geq 1$ let $\mathfrak{n}(t, f)$ be the number of poles of f, counting multiplicity, in $\{z : |z - it/2| \leq t/2, |z| \geq 1\}$, and set

$$\mathfrak{N}(r, f) = \int_1^r \frac{\mathfrak{n}(t, f)}{t^2} dt, \quad r \geq 1.$$

The Tsuji characteristic is defined as

$$\mathfrak{T}(r,f) = \mathfrak{m}(r,f) + \mathfrak{N}(r,f),$$

where

$$\mathfrak{m}(r,f) = \frac{1}{2\pi}\int_{\sin^{-1}(1/r)}^{\pi-\sin^{-1}(1/r)} \frac{\log^+|f(r\sin\theta e^{i\theta})|}{r\sin^2\theta}d\theta.$$

The upper half-plane is thus exhausted by circles of diameter $r \geq 1$ tangent to the real axis at 0. For non-constant f and any $a \in \mathbb{C}$ the first fundamental theorem then reads [5, 21]

$$\mathfrak{T}(r,f) = \mathfrak{T}(r, 1/(f-a)) + O(1), \quad r \to \infty, \tag{3}$$

and the lemma on the logarithmic derivative [15, p. 332] gives

$$\mathfrak{m}(r, f'/f) = O(\log r + \log^+ \mathfrak{T}(r,f)) \tag{4}$$

as $r \to \infty$ outside a set of finite measure. Further, $\mathfrak{T}(r,f)$ differs from a non-decreasing function by a bounded additive term [21]. Standard inequalities give

$$\mathfrak{T}(r, f_1+f_2) \leq \mathfrak{T}(r,f_1)+\mathfrak{T}(r,f_2)+\log 2, \quad \mathfrak{T}(r, f_1 f_2) \leq \mathfrak{T}(r,f_1)+\mathfrak{T}(r,f_2), \tag{5}$$

whenever f_1, f_2 are meromorphic in $\overline{H}$. Using the obvious fact that $\mathfrak{T}(r, 1/z) = 0$ for $r \geq 1$ we easily derive from (3) and (5) that $\mathfrak{T}(r,f)$ is bounded if f is a rational function.

A key role will be played by the following two results from [15]. The first is obtained by a change of variables in a double integral [15, p. 332].

Lemma 2.2 *Let $Q(z)$ be meromorphic in $\overline{H}$, and for $r \geq 1$ set*

$$m_{0\pi}(r, Q) = \frac{1}{2\pi}\int_0^\pi \log^+|Q(re^{i\theta})|d\theta. \tag{6}$$

Then for $R \geq 1$ we have

$$\int_R^\infty \frac{m_{0\pi}(r,Q)}{r^3}dr \leq \int_R^\infty \frac{\mathfrak{m}(r,Q)}{r^2}dr. \tag{7}$$

The second result from [15] is the analogue for the half-plane of Hayman's Theorem 3.5 from [7].

Lemma 2.3 *Let $k \in \mathbb{N}$ and let f be meromorphic in $\overline{H}$, with $f^{(k)} \not\equiv 1$. Then*

$$\mathfrak{T}(r,f) \leq \left(2+\frac{1}{k}\right)\mathfrak{N}\left(r,\frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\mathfrak{N}\left(r,\frac{1}{f^{(k)}-1}\right)+O(\log r+\log^{+}\mathfrak{T}(r,f))$$

as $r \to \infty$ outside a set of finite measure.

Lemma 2.3 is established by following Hayman's proof exactly, but using the Tsuji characteristic and the lemma on the logarithmic derivative (4).

We also need the following result of Yong Xing Gu (Yung-hsing Ku, [11]).

Lemma 2.4 *For every $k \in \mathbb{N}$, the meromorphic functions g in an arbitrary domain with the properties that $g(z) \neq 0$ and $g^{(k)}(z) \neq 1$ form a normal family.*

A simplified proof of this result is now available [23]. It is based on a rescaling lemma of Zalcman–Pang [17] which permits an easy derivation of Lemma 2.4 from the following result of Hayman: *Let $k \in \mathbb{N}$ and let g be a meromorphic function in the plane such that $g(z) \neq 0$ and $g^{(k)}(z) \neq 1$ for $z \in \mathbb{C}$. Then* $g = \text{const}$, see [6] or [7, Corollary of Thm 3.5].

3 Proof of Theorem 1.2

Let L, ψ, ϕ be as in the hypotheses, and assume that ϕ is transcendental but $L + L'/L$ has only finitely many non-real zeros. Condition (b) implies the Carathéodory inequality:

$$\frac{1}{5}|\psi(i)|\frac{\sin\theta}{r} < |\psi(re^{i\theta})| < 5|\psi(i)|\frac{r}{\sin\theta}, \quad r \geq 1, \quad \theta \in (0,\pi), \tag{8}$$

see, for example, [14, Ch. I.6, Thm 8′].

Lemma 3.1 *The Tsuji characteristic of L satisfies $\mathfrak{T}(r,L) = O(\log r)$ as $r \to \infty$.*

Proof. We apply Lemma 2.3 almost exactly as in [15, p. 334]. Let $g_1 = 1/L$. Then

$$g_1' = -L'/L^2.$$

Since L has only real poles and since $L + L'/L$ has by assumption finitely many non-real zeros it follows that g_1 and $g_1' - 1$ have finitely many zeros in H. Lemma 2.3 now gives $\mathfrak{T}(r,g_1) = O(\log r)$ initially outside a set of finite measure, and hence without exceptional set since $\mathfrak{T}(r,g_1)$ differs from a non-decreasing function by a bounded term. Now apply (3). □

Lemma 3.2 *The function ϕ has order at most* 1.

Proof. Again, this proof is almost identical to the corresponding argument in [15]. Lemmas 2.2 and 3.1 give

$$\int_R^\infty \frac{m_{0\pi}(r,L)}{r^3} dr \leq \int_R^\infty \frac{\mathfrak{m}(r,L)}{r^2} dr = O(R^{-1}\log R), \quad R\to\infty.$$

Since $m_{0\pi}(r,1/\psi) = O(\log r)$ by (8), we obtain using (2)

$$\int_R^\infty \frac{m_{0\pi}(r,\phi)}{r^3} dr = O(R^{-1}\log R), \quad R\to\infty.$$

But ϕ is entire and real on the real axis and so

$$\| \log^+ |\phi(re^{i\theta})| \| = 2m_{0\pi}(r,\phi).$$

Since $\| \log^+ |\phi(re^{i\theta})| \|$ is a non-decreasing function of r we deduce that

$$\| \log^+ |\phi(Re^{i\theta})| \| = O(R\log R), \quad R\to\infty,$$

which proves the lemma. $\square$

Lemma 3.3 *Let $\delta_1 > 0$ and $K > 1$. Then we have*

$$|wL(w)| > K, \quad |w| = r, \quad \delta_1 \leq \arg w \leq \pi - \delta_1, \tag{9}$$

for all r outside a set E_1 of zero logarithmic density.

Proof. Choose δ_2 with $0 < \delta_2 < \delta_1$. Let

$$\Omega_0 = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2, \quad \frac{\delta_2}{2} < \arg z < \pi - \frac{\delta_2}{2}\}.$$

For $r \geq r_0$, with r_0 large, let $g_r(z) = 1/(rL(rz))$. Then $g_r(z) \neq 0$ on Ω_0, since all poles of L are real. Further,

$$g_r'(z) = -L'(rz)/L(rz)^2.$$

Since L is analytic in H and $L + L'/L$ has finitely many zeros in H it follows that provided r_0 is large enough the equation $g_r'(z) = 1$ has no solutions in Ω_0. Thus the functions $g_r(z)$ form a normal family on Ω_0, by Lemma 2.4.

Suppose that $|w_0| = r \geq r_0$, and $\delta_1 \leq \arg w_0 \leq \pi - \delta_1$, and that

$$|w_0 L(w_0)| \leq K. \tag{10}$$

Then

$$|g_r(z_0)| \geq 1/K, \quad z_0 = \frac{w_0}{r},$$

and so since the g_r are zero-free and form a normal family we have

$$|g_r(z)| \geq 1/K_1, \quad |z| = 1, \quad \delta_2 \leq \arg z \leq \pi - \delta_2, \tag{11}$$

for some positive constant $K_1 = K_1(r_0, \delta_1, \delta_2, K)$, independent of r. By (2), (8), and (11) we have, for $|w| = r$, $\delta_2 \leq \arg w \leq \pi - \delta_2$, the estimates

$$|wL(w)| = |w\psi(w)\phi(w)| \leq K_1, \quad |\phi(w)| \leq K_2 = \frac{5K_1}{|\psi(i)| \sin \delta_2}. \tag{12}$$

Thus (10) implies (12). For $t \geq r_0$ let

$$E_2(t) = \{w \in \mathbb{C} : |w| = t, |\phi(w)| > K_2\}.$$

Further, let $\theta(t)$ be the angular measure of $E_2(t)$, and as in Lemma 2.1 let $\theta^*(t) = \theta(t)$, except that $\theta^*(t) = \infty$ if $E_2(t) = C(0,t)$. Let

$$E_3 = \{t \in [r_0, \infty) : \theta(t) \leq 4\delta_2\}.$$

Since (10) implies (12), we have (9) for $t \in [r_0, \infty) \setminus E_3$. Applying Lemma 2.1 we obtain, since ϕ has order at most 1 by Lemma 3.2,

$$(1 + o(1)) \log r \geq \int_{r_0}^{r} \frac{\pi dt}{t\theta^*(t)} \geq \int_{[r_0,r] \cap E_3} \frac{\pi dt}{4\delta_2 t},$$

from which it follows that E_3 has upper logarithmic density at most $4\delta_2/\pi$. Since δ_2 may be chosen arbitrarily small, the lemma is proved. $\square$

The estimates (8) and (9) and the fact that ϕ is real now give

$$|\phi(z)| > \frac{K \sin \delta_1}{5|\psi(i)|r^2}, \quad \delta_1 \leq |\arg z| \leq \pi - \delta_1,$$

for $|z| = r$ in a set of logarithmic density 1. Since ϕ has order at most 1 but is transcendental, we deduce (compare [9, pp. 500-501]):

Lemma 3.4 *The function ϕ has infinitely many zeros.* □

Let

$$F(z) = z - \frac{1}{L(z)}, \quad F'(z) = 1 + \frac{L'(z)}{L(z)^2}. \tag{13}$$

Since L has only real poles and $L + L'/L$ has finitely many non-real zeros we obtain at once:

Lemma 3.5 *The function F has finitely many critical points over $\mathbb{C} \setminus \mathbb{R}$, i.e. zeros z of F' with $F(z)$ non-real.* □

Lemma 3.6 *There exists $\alpha \in H$ with the property that $F(z) \to \alpha$ as $z \to \infty$ along a path γ_α in H.*

Lemma 3.6 is a refinement of Theorem 4 of [20], and will be proved in §4.

Now set

$$g(z) = z^2 L(z) - z = \frac{zF(z)}{z - F(z)}, \quad h(z) = \frac{1}{F(z) - \alpha}, \tag{14}$$

in which α is as in Lemma 3.6. Then g is analytic in $H \cup \{0\}$ and (5), (13) and Lemma 3.1 give

$$\mathfrak{T}(r, g) + \mathfrak{T}(r, h) = O(\log r), \quad r \to \infty.$$

Hence Lemma 2.2 leads to

$$\int_1^\infty \frac{m_{0\pi}(r, g)}{r^3} dr + \int_1^\infty \frac{m_{0\pi}(r, h)}{r^3} dr < \infty, \tag{15}$$

in which $m_{0\pi}(r, g)$ and $m_{0\pi}(r, h)$ are as defined in (6).

Lemma 3.7 *The function F has at most four finite non-real asymptotic values.*

Proof. Assume the contrary. Since $F(z)$ is real on the real axis we may take distinct finite non-real $\alpha_0, \ldots, \alpha_n$, $n \geq 2$, such that $F(z) \to \alpha_j$ as $z \to \infty$ along a simple path $\gamma_j : [0, \infty) \to H \cup \{0\}$. Here we assume that $\gamma_j(0) = 0$, that $\gamma_j(t) \in H$ for $t > 0$, and that $\gamma_j(t) \to \infty$ as $t \to \infty$. We may further assume that $\gamma_j(t) \neq \gamma_{j'}(t')$ for $t > 0$, $t' > 0$, $j \neq j'$.

Re-labelling if necessary, we obtain n pairwise disjoint simply connected domains $D_1, \ldots, D_n$ in H, with D_j bounded by γ_{j-1} and γ_j, and for $t > 0$ we let $\theta_j(t)$ be the angular measure of the intersection of D_j with the circle $C(0,t)$. By (14), the function $g(z)$ tends to α_j as $z \to \infty$ on γ_j, and so $g(z)$ is unbounded on each D_j but bounded on the finite boundary ∂D_j of each D_j. Let c be large and positive, and for each j define

$$u_j(z) = \log^+ |g(z)/c|, \quad z \in D_j. \tag{16}$$

Set $u_j(z) = 0$ for $z \notin D_j$. Then u_j is continuous, and subharmonic in the plane since g is analytic in $H \cup \{0\}$.

Lemma 2.1 gives, for some $R > 0$ and for each j,

$$\int_R^r \frac{\pi dt}{t\theta_j(t)} \leq \log \|u_j(4re^{i\theta})\| + O(1)$$

as $r \to \infty$. Since u_j vanishes outside D_j we deduce using (16) that

$$\int_R^r \frac{\pi dt}{t\theta_j(t)} \leq \log m_{0\pi}(4r, g) + O(1), \quad r \to \infty, \tag{17}$$

for all $j \in \{1, \ldots, n\}$. However, the Cauchy-Schwarz inequality gives

$$n^2 \leq \sum_{j=1}^n \theta_j(t) \sum_{j=1}^n \frac{1}{\theta_j(t)} \leq \sum_{j=1}^n \frac{\pi}{\theta_j(t)}$$

which on combination with (17) leads to, for some positive constant c_3,

$$n \log r \leq \log m_{0\pi}(4r, g) + O(1), \quad m_{0\pi}(r, g) \geq c_3 r^n, \quad r \to \infty.$$

Since $n \geq 2$ this contradicts (15), and Lemma 3.7 is proved. $\square$

From Lemmas 3.5 and 3.7 we deduce that the inverse function F^{-1} has finitely many non-real singular values. Using Lemma 3.6, take $\alpha \in H$ such that $F(z) \to \alpha$ along a path γ_α tending to infinity in H, and take ε_0 with $0 < \varepsilon_0 < \text{Im}(\alpha)$ such that F has no critical or asymptotic values in $0 < |w - \alpha| \leq \varepsilon_0$. Take a component C_0 of the set $\{z \in \mathbb{C} : |F(z) - \alpha| < \varepsilon_0\}$ containing an unbounded subpath of γ_α. Then by a standard argument [16, XI.1.242] involving a logarithmic change of variables the inverse function F^{-1} has a logarithmic singularity over α, the component C_0 is simply connected, and $F(z) \neq \alpha$ on C_0. Further, the boundary of C_0 consists of a single simple curve going to infinity in both directions. Thus we may define a continuous, non-negative, non-constant subharmonic function in the plane by

$$u(z) = \log \left| \frac{\varepsilon_0}{F(z) - \alpha} \right| = \log |\varepsilon_0 h(z)| \quad (z \in C_0), \qquad u(z) = 0 \quad (z \notin C_0), \tag{18}$$

using (14).

The next lemma follows from (13) and (18).

Lemma 3.8 *For large z with $|zL(z)| > 3$ we have $|F(z) - \alpha| > |z|/2$ and $u(z) = 0$.* □

Lemma 3.9 *We have*

$$\lim_{r \to \infty} \frac{\log \|u(re^{i\theta})\|}{\log r} = \infty. \tag{19}$$

Proof. Apply Lemma 3.3, with $K = 3$ and δ_1 small and positive. By Lemma 3.8 we have $u(z) = 0$ if $\delta_1 \leq |\arg z| \leq \pi - \delta_1$ and $|z|$ is large but not in E_1. For large t let $\sigma(t)$ be the angular measure of that subset of $C(0, t)$ on which $u(z) > 0$. Since u vanishes on the real axis Lemma 2.1 and Lemma 3.3 give, for some $R > 0$,

$$\log \|u(4re^{i\theta})\| + O(1) \geq \int_R^r \frac{\pi dt}{t\sigma(t)} \geq \int_{[R,r] \setminus E_1} \frac{\pi dt}{4\delta_1 t} \geq \frac{\pi}{4\delta_1}(1 - o(1)) \log r$$

as $r \to \infty$. Since δ_1 may be chosen arbitrarily small the lemma follows. □

Now (18) and the fact that u vanishes outside C_0 give

$$\|u(re^{i\theta})\| \leq m_{0\pi}(r, h) + O(1),$$

from which we deduce using (19) that

$$\lim_{r \to \infty} \frac{\log m_{0\pi}(r, h)}{\log r} = \infty.$$

This obviously contradicts (15), and Theorem 1.2 is proved. □

4 Proof of Lemma 3.6

The proof is based essentially on Lemmas 1 and 5 and Theorem 4 of [20]. Assume that there is no $\alpha \in H$ such that $F(z)$ tends to α along a path tending to infinity in H.

Let

$$W = \{z \in H : F(z) \in H\}, \quad Y = \{z \in H : L(z) \in H\}.$$

Then $Y \subseteq W$, by (13), so that each component C of Y is contained in a component A of W.

Lemma 4.1 *To each component A of W corresponds a finite number $v(A)$ such that F takes every value at most $v(A)$ times in A and has at most $v(A)$ distinct poles on ∂A.*

Proof. Using Lemma 3.5, cut H along a simple polygonal curve starting from 0 so that the resulting simply connected domain D contains no critical value of F. Let $X = \{z \in H : F(z) \in D\}$. By analytic continuation of the inverse function, every component B of X is simply connected and conformally equivalent under F to D. Moreover, if the finite boundary ∂B contains no critical point z of F with $F(z) \in H$ then the branch of the inverse function mapping D onto B may be analytically continued throughout H, and in this case F is univalent in the component A of W containing B.

Fix a component A of W on which F is not univalent. If $z \in A$ with $F(z) \in D$ then z lies in a component $B \subseteq A$ of X, such that ∂B contains a critical point of F. Since F has finitely many critical points over H, and every critical point can belong to the boundaries of at most finitely many components B of X, it follows that A contains finitely many such components B. Application of the open mapping theorem gives us a finite $v(A)$ satisfying the first statement of the lemma.

If $z_0 \in \partial A$ is a pole of F, then for an arbitrarily small neighbourhood N of z_0, F assumes in $N \cap A$ all sufficiently large values in H. It follows that there are no more than $v(A)$ distinct poles of F on ∂A. $\square$

Lemma 4.2 *There are infinitely many components A of W such that A contains an unbounded component C of Y and $\partial A \cap \partial C$ contains a zero of L.*

Proof. We first note [20, Lemma 1] that L has no poles in the closure of Y. To see this, let x_0 be a pole of L. Then x_0 is real and is a simple pole of L with positive residue. Hence $\lim_{y\to 0+} \mathrm{Im}(L(x_0+iy)) = -\infty$ and since L is univalent on an open disc $N_0 = B(x_0, R_0)$ it follows that $\mathrm{Im}(L(z)) < 0$ on $N_0 \cap H$.

Thus every component C of Y is unbounded by the maximum principle, since $\mathrm{Im}(L(z))$ is harmonic in H and vanishes on ∂C.

Next, we recall from Lemma 3.4 that ϕ has infinitely many zeros; by the hypotheses (2) and (c) these must be zeros of L. Suppose first that

(I) L has infinitely many non-real zeros.

Then L has infinitely many zeros $\eta \in H$, and $F(\eta) = \infty$. Each such η lies on the boundary of a component C of Y, and C is contained in a component A of W, and $\eta \in \partial A$. By Lemma 4.1 we obtain in this way infinitely many components A.

Since L is real on the real axis we obtain the same conclusions if either of the following conditions hold:

(II) L has infinitely many multiple zeros;

(III) L has infinitely many real zeros x with $L'(x) > 0$.

We assume henceforth that neither (I) nor (II) holds, and will deduce (III). Let $\{a_k\}$ denote the poles of L, in increasing order. Then there are two possibilities. The first is that there exist infinitely many intervals (a_k, a_{k+1}) each containing at least one zero x_k of ϕ. Since ψ must have negative residues by (b) there must be a zero y_k of ψ in (a_k, a_{k+1}), and we may assume that $y_k \neq x_k$, since L has by assumption finitely many multiple zeros. But then the graph of L must cut the real axis at least twice in (a_k, a_{k+1}), and so there exists a zero x of L in (a_k, a_{k+1}) with $L'(x) > 0$. Thus we obtain (III).

The second possibility is that we have infinitely many pairs of zeros a, b of ϕ such that L has no poles on $[a, b]$. In this case we again obtain a zero x of L with $L'(x) > 0$, this time in $[a, b]$, and again we have (III). $\square$

We now complete the proof of Lemma 3.6. Combining Lemmas 3.5, 4.1 and 4.2 we obtain at least one zero η of L, with $\eta \in \partial A \cap \partial C$, in which A, C are components of W, Y respectively, C is unbounded and $C \subseteq A$, and F has no critical point in A. It then follows by analytic continuation of the inverse function in H that F maps A univalently onto H. Since F takes near η all values w of positive imaginary part and large modulus, it follows that $F(z)$ is bounded as $z \to \infty$ in A, so that $L(z) \to 0$ as $z \to \infty$ in A, and hence as $z \to \infty$ in C. This contradicts the maximum principle. $\square$

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