Math 530

Practice problems

1. What is the radius of convergence of the power series centered at zero for the function $1/(z-1-i)^{10}$?

2. Prove that power series can be integrated term by term. To be precise, suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$ converges on the disc $D_R(0)$ to an analytic function $f(z)$. Prove that the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ also has radius of convergence R and that this series converges to an analytic anti-derivative of $f(z)$ inside the circle of convergence.

3. Suppose that f and g are analytic in a neighborhood of a. If f has a simple zero at a, then

$$\mathrm{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}.$$

Prove a similar formula in case f has a double zero at a, i.e., in case f is such that $f(a) = 0$, $f'(a) = 0$, but $f''(a) \neq 0$.

4. Show that if f is an analytic mapping of the unit disk into itself such that $f(a) = 0$, then

$$|f(z)| \leq \left| \frac{z-a}{1-\bar{a}z} \right|$$

for all z in the disk.

5. Show that if f is an analytic mapping of the unit disk into itself, then $|f'(0)| \leq 1$.

6. Suppose that f is an analytic function on the unit disc such that $|f(z)| < 1$ for $|z| < 1$. Prove that if f has a zero of order n at the origin, then

$$|f(z)| \leq |z|^n$$

for $|z| < 1$. How big can $|f^{(n)}(0)|$ be?

7. Suppose that f is an entire function that satisfies an estimate $|f(z)| \leq C(1 + |z|^N)$ for all z where C is a positive constant and N is a positive integer. Prove that f must be a polynomial of degree N or less.

8. Suppose f and g are analytic functions on a domain Ω and that

$$\sin(f(z)) \equiv \sin(g(z))$$

on Ω. How must f and g be related?