Equation $\Delta u + e^{2u} = 0$ in the plane

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The differential equation

$$\Delta u + e^{2u} = 0 \tag{1}$$

in a simply connected plane region has the general solution

$$u = \log \frac{2|f'|}{1+|f|^2} \tag{2}$$

where f is a meromorphic locally univalent function in this region. Locally univalent is the same as local homeomorphism, and for a meromorphic functions it is equivalent to the condition that $f'(z) \neq 0$ and all poles of f are simple.
This form of the solution was discovered by J. Liouville in 1850. An equivalent fact is that every two Riemannian metrics of constant curvature are locally isometric is contained in the paper of F. Minding (1839).

The metric σ related to our equation is defined by its length element $e^{2u(z)}|dz|$, and the meromorphic function f in (2) is called the *developing map.* So our metric σ is the pull back of the standard spherical metric with line element

$$\frac{2|dz|}{1+|z|^2}.$$

The expression

$$f^{\#} = \frac{2|f'|}{1+|f|^2}$$

is called the *spherical derivative* of the meromorphic function f. The purpose of this talk is to explain how the theory of meromorphic functions can answer questions about equation (1).

We begin with introducing the Nevanlinna characteristic of a meromorphic function, which is an analog for transcendental functions of the degree of a rational function. For a meromorphic function in the plane we define

$$A(r,f)=\frac{1}{\pi}\int_{|z|\leq r}\frac{|f'|^2}{1+|f|^2}dxdy=\frac{1}{4\pi}\int_{|z|\leq r}e^{2u}dxdy.$$

This is the ratio of the area of the disk $|z|\leq r$ with respect to σ to the area of the unit sphere, which can be interpreted as an average covering number of the sphere by the image of the disk. Then it is useful to take an average in r:

$$T(r,f)=\int_0^r\frac{A(t,f)}{t}dt.$$

This is called the *Nevanlinna characteristic* of f. One advantage of this definition is that $T(r,f)$ has nicer properties than $A(r,f)$. We list some of these properties.

1. For a non-constant f, $T(r, f)$ is increasing, tends $+\infty$, and is convex with respect to the logarithm ($rT'(r) \geq 0$). When f is rational of degree d

$$T(r, f) = (d + o(1)) \log r,$$

while for every transcendental function f

$$\frac{T(r, f)}{\log r} \to \infty.$$

2. $T(r, fg) \leq T(r, f) + T(r, g) + O(1)$,

$$T(r, f + g) \leq T(r, f) + T(r, g) + O(1),$$

$$T(r, 1/f) = T(r, f), \quad T(r, f^2) = 2T(r, f) + O(1).$$

These properties show that $T(r, f)$ is a generalization of degree.

There is an alternative definition. Let $n(t)$ be the number of poles of f, counting multiplicity, in $|z| \leq t$. We define

$$N(r,f) = \int_0^r \frac{n(t) - n(0)}{t} dt + n(0) \log r,$$

$$m(r,f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log +|f(re^{i\theta})| d\theta,$$

Then

$$T(r,f) = m(r,f) + N(r,f) + O(1),$$

from which the second set of properties follows.
The main technical tool of Nevanlinna theory is the Lemma on the logarithmic derivative:

$$m(r, f'/f) = O(\log T(r,f) + \log r), \quad r \to \infty, r \notin E$$

where E is a set of finite measure.

These elementary properties immediately imply the famous theorem of Chen and Li (1991):

If the area of σ is finite then the developing map is linear-fractional.

Indeed, a locally univalent rational function must be of degree 1. Once this is established, a direct computation gives that all such solutions are radially symmetric with respect to some center, and other properties.

In this talk, I'll explain several other results on equation (1) which were recently obtained by Changfeng Gui, Qinfeng Li, Lu Xu, Walter Bergweiler, Jim Langley and myself.

Let us impose a growth restriction on solutions: we will say that $u \in N(k)$ if

$$\limsup_{z \to \infty} \frac{u(z)}{\log |z|} = k < \infty, \quad N := \bigcup_{k < +\infty} N(k).$$

Theorem 1. Class $N(k)$ is non-empty if and only if $k = -2$ or $2k$ is a non-negative integer. For solutions in $N(-2)$, function f is linear-fractional, while for solutions in $N(0)$, f is an exponential, and for all $k \geq 0$ we have the following asymptotics:

$$f^{\#}(re^{i\theta}) = -(c + o(1))r^{k+1} |\sin(k+1)(\theta - \theta_0)|,$$

with some $c > 0$ and $\theta_0 \in \mathbf{R}$.

In particular, this theorem gives a classification of all bounded from above solutions and explicit formulas for them:

Solutions bounded from above belong to $N(0) \cup N(-2)$, so the corresponding developing map is either linear-fractional or an exponential.

It follows from the asymptotic formula that a solution of class $N(k)$ cannot be concave, except when $k \leq 0$. It can be verified directly that solutions in $N(-2)$ are not concave.

One can generalize this result. A function u is called *quasiconcave* if for every segment $[a, b]$ and a point $c \in [a, b]$ we have

$$u(c) \geq \min\{u(a), u(b)\}.$$

Theorem 2. *For quasiconcave solutions of equation (1) the developing map is either exponential of linear-fractional.*

For solutions of class $N(k), \ k < \infty$ this follows from the asymptotic formula above, but extending this to all solutions of (1) requires more complicated tools.

The union of classes $N(k)$ corresponds to developing maps of *finite order*, where the order ρ of a meromorphic function is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

so the order of a function $f \in N(k)$ is $k+1$ when $k \geq 0$. Theorem 1 is an immediate consequence of the results of R. Nevanlinna (1932), and I will sketch a proof.
Consider the *Schwarzian derivative* of f:

$$P = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{3}$$

When f is meromorphic and locally univalent, P has no poles, so it is an entire function. On the other hand, the assumption that f is of finite order permits to estimate the growth of P, using the Lemma on the logarithmic derivative, and one obtains $T(r,P) = O(\log r)$.

So P must be a polynomial, and we can consider (3) as a differential equation for f.
Now it easy to verify by computation that the general solution of the Schwarz differential equation is a ratio of two linearly independent solutions of the linear ODE

$$w'' + (P/2)w = 0,$$

and for this linear ODE there is a powerful asymptotic theory which gives asymptotic expansions of solutions. This implies Theorem 1.

For the proof of Theorem 2, we still have that $f = w_1/w_2$ is a ratio of two solutions of a linear differential equation

$$w'' + Aw = 0, \tag{4}$$

as before, but since no growth estimate is available, A is an *arbitrary* entire function, so little can be said about its asymptotic behavior.

Nevertheless, there is a theory due to Wiman and Valiron, which gives some control on the asymptotic behavior of an arbitrary entire function near the points z_r where its maximum modulus is attained:

$$|f(z_r)| = \max_{|z|=r} |f(z)|.$$

This theory tells us that in certain neighborhood of such a point we have

$$f(z) = (1 + o(1)) \left(\frac{z}{z_r} \right)^{n(r)} f(z_r),$$

where $n(r)$ is the *central index*: the index of the maximal term of the Taylor series at $|z| = r$. This result allows an approximate asymptotic integration of equation (4).

So the proof of Theorem 2 uses asymptotic integration near the point where the maximum modulus of A is attained.

Next I address the questions of Changfeng Gui and Qinfeng Li about the diameter of the plane with the metric σ whole line element is $e^u|dz|$. I recall that the distance between two points is defined as an infimum of lengths of curves connecting these points and the length of a curve γ is

$$|\gamma| = \int_\gamma \frac{2|f'|}{1+|f|^2}|dz|,$$

where f is the developing map. This length is equal to the length of the image $f(\gamma)$ in the standard metric on the sphere.

Since a con-constant meromorphic function in the plane has a dense image, the diameter of the metric is at least π, and this lower bound is attained when f is linear-fractional. It also attained for the case when $f(z) = e^z$. Changfeng Gui and Qinfeng Li computed the diameter for the case $f(z) = e^z + t, t \geq 0$ and it is equal to

$$\pi + 2 \arctan t.$$

These results led them to conjecture that for all other cases (that is when f is neither linear-fractional nor exponential) the diameter is strictly greater than π.

This conjecture is confirmed by the following

Theorem 3. The diameter of the plane with metric σ is at least $4\pi/3$, unless the developing map f is linear-fractional or exponential.

We could not determine whether this lower bound $4\pi/3$ is optimal. For the proof of Theorem 3, I recall definitions and facts related to singularities of the inverse function f^{-1}.
Consider germs ϕ of f^{-1} and its analytic continuation along some curve $\Gamma : [0, 1) \to \overline{\mathbf{C}}$. If the endpoint $a = \gamma(1) \in \overline{\mathbf{C}}$ of this curve is a singularity, then $\gamma(t) = \phi(\Gamma(t)) \to \infty$. So to each singularity corresponds an *asymptotic curve* γ with the property that $f(\gamma(t))$ has a limit $a \in \overline{\mathbf{C}}$ while $\gamma(t) \to \infty, t \to 1$.
If there exists such a curve, then a is called an *asymptotic value.*

Suppose that some region $D \subset \overline{\mathbf{C}}$ contains no asymptotic values. Then the restriction to each component:

$$f : \text{a component of } f^{-1}(D) \to D$$

is a covering map. In particular, when D is simply connected, this restriction is a homeomorphism.

Let $a \in \overline{\mathbf{C}}$ be an asymptotic value of a locally univalent meromorphic function f. Let $\mathfrak{A}$ be the set of all connected neighborhoods of a. Consider a function V which assigns to every $O \in \mathfrak{A}$ a component of $f^{-1}(O)$ such that

$$O_1 \subset O_2 \to V(O_1) \subset V(O_2), \quad \text{and} \quad \bigcap_{O \in \mathfrak{A}} V(O) = \emptyset.$$

Each such function V is called a singularity of f^{-1} over a. The sets $V(O)$ are called neighborhoods of the singularity. Every neighborhood of a singularity over a contains an asymptotic curve with asymptotic value a.

A singularity is called *isolated* if it has a neighborhood which is not a neighborhood of any other singularity.

The proof of Theorem 3 consists of two parts.

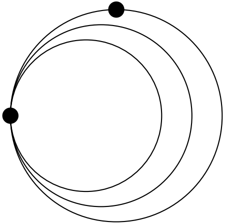
1. f^{-1} has a non-isolated singularity. In this case one can show that there exists a curve $\gamma : (-1, 1) \subset \mathbf{C}$ which is doubly asymptotic, that is $\gamma(t) \to \infty,\ t \to \pm 1$ and the two limits $\lim_{t \to \pm 1} f(\gamma(t))$ exist and are distinct. Moreover, one can find such a double asymptotic curve of an arbitrarily small diameter ϵ. Such a curve γ splits the plane into two parts D_1 and D_2, We claim that the restrictions of f on D_j have dense images.
This follows from a theorem of Lindelöf:

A bounded analytic function in the unit disk cannot have two distinct asymptotic values for asymptotic curves ending at the same point of the unit circle.

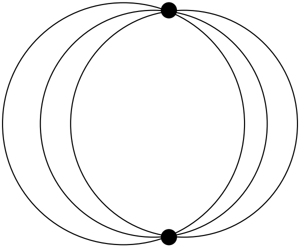
Since D_1 and D_2 are simply connected we can map them on the unit disk, and infinity will correspond to one point on the unit circle. So we can apply Lindelöf's theorem: assuming that $f(D_j)$ omits a neighborhood of some point $a \in \overline{\mathbf{C}}$ we consider the function $1/(f - a)$ in D_i and obtain a contradiction.

So we can find points $z_j \in D_j$ such that the distance from $f(z_j)$ to $f(\gamma)$ is at least $\pi - 2\epsilon$, and then, since every curve from z_1 to z_2 must cross γ, we obtain that the diameter of the plane with metric σ is at least $2\pi - 4\epsilon$.

2. All singularities of f^{-1} are isolated. In this case,
We can find a disk in $\overline{\mathbf{C}}$ *in which a branch of* f^{-1} *exists, and which has at least three singularities on its boundary.*
Indeed, take an arbitrary branch of f^{-1} and consider the largest disk where it is holomorphic. Then the boundary of this disk contains at least one singularity.
If there is only one, we can move the center of the disk away from this only singularity, and obtain larger disks (containing the original disk and all its boundary except the asymptotic value).



This is the place where we use the assumption that singularities are isolated.
Eventually we find the disks B_0 with two singularities on the boundary. Suppose that there are only two of them.
Let β be an arc of a great circle through the center of B_0 equidistant from the two singularities. We can move the center of B_0 along this arc, and obtain a family of disks B_t with the same two singularities on the boundary.



If this procedure stops for some $t \in \mathbf{R}$, we obtain a disk with at least 3 singularities on the boundary. If it does not stop, then one can show that f is a universal cover over $\overline{\mathbf{C}}$ minus two points, that is an exponential function.
So, if f is not an exponential, we obtain a disk in $\overline{\mathbf{C}}$ where a holomorphic branch of f^{-1} exists, and which has three singularities on the boundary. So there are two distinct asymptotic values at the distance $\leq 2\pi/3$ between them. Considering the shortest curve between these asymptotic values, we obtain a curve γ in $\mathbf{C}$ both ends of which tend to ∞, and such that $f(\gamma)$ is a geodesic segment of length at most $2\pi/3$. This curve divides the plane into two regions D_1 and D_2, and the restrictions of f on these regions has dense image, so the proof is completed in the same way as in Case 1:

If we have a geodesic arc Γ of length at most $2\pi/3$ in $\overline{\mathbf{C}}$, then there is a point at the distance at lest $2\pi/3$ from this curve.

Since the images $f(D_1)$ and $f(D_2)$ are dense, we can find two points z_1, z_2 in $\mathbf{C}$ such that both $f(z_1)$ and $f(z_2)$ are at the distance at least $2\pi/3$ from $f(\gamma)$. Then the image of any curve connecting z_1 and z_2 must go from $f(z_1)$ to $f(\gamma)$ and then to $f(z_2)$, so this image has length at least $4\pi/3$.
This completes the proof of Theorem 3.

We could not determine whether our estimate $4\pi/3$ is best possible. The natural candidate for diameter $2\pi/3$ will be a ratio of two Airy functions, $f = w_1/w_2$, where w_j are appropriate solutions of

$$w'' + zw = 0$$

One can choose these solutions so that f^{-1} has three singularities with asymptotic values at the cube roots of unity.
It should not be too hard to compute the diameter of the corresponding Riemann surface spread over the sphere exactly, but we did not do this.

Final remark. For functions $u \in N(k)$ the diameter of the metric σ is bounded from above by a constant that depends only on k. However one can construct a function f (of infinite order) for which this diameter is infinite. The simplest example of such a function is a ratio $f = w_1/w_2$ of two solutions of the Mathieu equation

$$w'' + (\cos z + \lambda)\, w = 0,$$

for some special choice of parameter λ.
The proof is based on the analysis of the Riemann surface spread over the sphere corresponding to f^{-1}. One can describe this Riemann surface geometrically (by pasting together certain spherical triangles), and the Mathieu equation is used to sow that it is of parabolic type.