# Analytic continuation of eigenvalues of a quartic oscillator

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**Abstract**

We consider the Schrödinger operator on the real line with even quartic potential $x^4+\alpha x^2$ and study analytic continuation of eigenvalues, as functions of parameter $\alpha$. We prove several properties of this analytic continuation conjectured by Bender, Wu, Loeffel and Martin. 1. All eigenvalues are given by branches of two multi-valued analytic functions, one for even eigenfunctions and one for odd ones. 2. The only singularities of these multi-valued functions in the complex $\alpha$-plane are algebraic ramification points, and there are only finitely many singularities over each compact subset of the $\alpha$-plane.

## 1. Introduction.

Consider the boundary value problem on the real line:

$$-y''+(\beta x^4+x^2)y=\lambda y, \quad y(-\infty)=y(\infty)=0. \tag{1}$$

If $\beta>0$, then this problem is self-adjoint, it has a discrete spectrum of the form $\lambda_0<\lambda_1<\ldots\to+\infty$, and every eigenspace is one-dimensional. The eigenvalues $\lambda_n$ are real analytic functions of $\beta$ defined on the positive ray.

In 1969, Bender and Wu [5] studied analytic continuation of $\lambda_n$ to the complex $\beta$-plane. Their main discoveries are the following:

(i) For every non-negative integers $m$ and $n$ of the same parity, the function $\lambda_m$ can be obtained by an analytic continuation of the function $\lambda_n$ along some path in the complex $\beta$-plane.

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(ii) The only singularities encountered in the analytic continuation of $\lambda_n$ in the punctured $\beta$-plane $\mathbf{C}\backslash\{0\}$ are algebraic ramification points.

(iii) These ramification points accumulate to $\beta = 0$, in such a way that no analytic continuation of any $\lambda_n$ to 0 is possible.

The last statement gives a good reason why the formal perturbation series of $\lambda_n$ in powers of $\beta$ is divergent. Bender and Wu also studied the global structure of the Riemann surfaces of the functions $\lambda_n$ spread over the $\beta$-plane, that is the position of their ramification points, and how the sheets of these Riemann surfaces are connected at these points.

Bender and Wu used a combination of mathematical and heuristic arguments with numerical computation. Since the publication of their paper, several results about analytic continuation of $\lambda_n$ were proved rigorously. We state some of these results. A broader context in which this problem arises is described in the survey [34].

First of all, we recall a change of the variable which substantially simplifies the problem [32]. Consider the family of differential equations

$$H(\alpha,\beta)y = \lambda y, \quad \text{where} \quad H(\alpha,\beta) = -d^2/dx^2 + (\beta x^4 + \alpha x^2).$$

The change of the independent variable

$$w(x) = y(tx) \tag{2}$$

gives the differential equation $H(t^4\alpha, t^6\beta)w = t^2\lambda w$. Thus, if $\alpha$ and $t$ are real, and $\beta > 0$, we have

$$\lambda_n(t^4\alpha, t^6\beta) = t^2\lambda_n(\alpha,\beta).$$

If $\alpha = 1$ we can take $t = \beta^{-1/6} > 0$ and obtain

$$\lambda_n(1,\beta) = \beta^{1/3}\lambda_n(\beta^{-2/3}, 1). \tag{3}$$

This reduces our problem to the study of the analytic continuation of eigenvalues of the one-parametric family

$$-y'' + (x^4 + \alpha x^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0, \tag{4}$$

depending on the complex parameter $\alpha$. The study of this family of quartic oscillators is equivalent to the study of the family (1). Indeed, if we know an analytic continuation of $\lambda_n(\alpha, 1)$ along some curve in the $\alpha$-plane, then equation (3) gives an analytic continuation of $\lambda_n(1, \beta)$ in the $\beta$-plane and vice versa. The main advantage of restating the problem in the form (4) is that any analytic continuation of an eigenvalue (eigenfunction) of (4) remains an eigenvalue (eigenfunction) of (4). This is not so for the operator in (1): when we perform an analytic continuation of an eigenfunction the result may no longer be an eigenfunction, because it may fail to satisfy the boundary condition, see [32], [4].

*From now on we consider only the family (4), and slightly change our notation: the $\lambda_n$ will be real analytic functions of $\alpha > 0$ representing the eigenvalues of (4).*

Notice that the $\lambda_n$ have immediate analytic continuations[1] from the positive ray to the whole real line.

Loeffel and Martin [24] proved that the functions $\lambda_n$ have immediate analytic continuations to the sector $|\arg \alpha| < 2\pi/3$. They conjectured that the radius of convergence of the power series of $\lambda_n$ at $\alpha = 0$ tends to infinity as $n \to \infty$.

Simon [32, 33] proved that the singularities of the $\lambda_n$ accumulate to $\infty$ in the asymptotic direction of the negative ray. More precisely, for every $n$ and for every $\eta \in (0, \pi)$, there exists $B = B_n(\eta)$ such that $\lambda_n$ has an immediate analytic continuation from the positive ray to the region $\{\alpha : |\arg \alpha| < \eta, |\alpha| > B\}$. On the other hand, he also proved that the $\lambda_n$ do not have immediate analytic continuations to full punctured neighborhoods of $\infty$. This proves statement (iii) of Bender and Wu and implies divergence of the perturbation series for $\lambda_n$ at $\beta = 0$.

Delabaere, Dillinger and Pham in their interesting papers [7, 8, 9] used a version of the WKB method to study operator (4) for large $\alpha$. They claim to confirm all conclusions of Bender and Wu, however it is not clear to us what is proved rigorously in [7, 8, 9], which statements are heuristic and which are verified numerically. In particular, we could not determine whether these papers contain a complete proof of the statement (i).

Another study of $\lambda_n$ for large $\alpha$ is [20]. It is not clear whether statements of Theorem 1 can be derived from the results of [20].

[1] We say that a function analytic on a set $X \subset \overline{\mathbf{C}}$ has an immediate analytic continuation to a set $Y \subset \overline{\mathbf{C}}$ if $X \cap Y$ has limit points in $X$ and there exists an analytic function $g$ on $Y$ such that $f(z) = g(z)$ for $z \in X \cap Y$.

In this paper we give complete proofs of statements (i) and (ii) of Bender and Wu and of the conjecture of Loeffel and Martin. Our methods are different from those of all papers mentioned above.

**Theorem 1.**
a) *All* $\lambda_n$ *are branches of two multi-valued analytic functions* $\Lambda^i$, $i = 0, 1$, *of* $\alpha$, *one for even* $n$ *another for odd* $n$.

b) *The only singularities of* $\Lambda^i$ *over the* $\alpha$*-plane are algebraic ramification points.*

c) *For every bounded set* $X$ *in the* $\alpha$*-plane, there are only finitely many ramification points of* $\Lambda^i$ *over* $X$.

Statements a) and b) prove (i) and (ii) of Bender and Wu. Statement c) implies the conjecture of Loeffel and Martin stated above.

Statements b) and c) actually hold in greater generality. Let $P(a, z) = z^d + a_{d-1}z^{d-1} + \ldots + a_1 z$ be any monic polynomial of even degree $d$, with complex coefficients $a = (a_1, \ldots, a_{d-1}) \in \mathbf{C}^{d-1}$. Consider the boundary value problem

$$-y'' + P(a, z)y = \lambda y, \tag{5}$$

$$y(+\infty) = y(-\infty) = 0, \tag{6}$$

where the boundary condition is imposed on the real axis. For non-real $a$, this problem is not self-adjoint, however it is known [31] that the spectrum of this problem is infinite and discrete, eigenspaces are one-dimensional, and the eigenvalues tend to infinity in the asymptotic direction of the positive ray. Let $\Lambda$ be the multi-valued function of $a \in \mathbf{C}^{d-1}$ which to every $a$ puts into correspondence the set of eigenvalues of (5), (6). We write the set $\Lambda(a)$ as $\Lambda(a) = \{\mu_0(a), \mu_1(a), \ldots\}$ where

$$|\mu_0(a)| \leq |\mu_1(a)| \leq \ldots \to +\infty.$$

So for real $a$ we have $\mu_n(a) = \lambda_n(a)$ but this does not have to be the case for complex $a$. (The functions $\mu_n$ are not expected to be analytic; they are only piecewise analytic). It is known [31] that there exists an entire function $F$ of $d$ variables with the property that $\lambda$ is an eigenvalue of the problem (5), (6) if and only if

$$F(a, \lambda) = 0. \tag{7}$$

Equation (7) may be called the characteristic equation of the problem (5), (6). It is the equation implicitly defining our multi-valued function $\Lambda$.

**Theorem 2.** *The only singularities of* $\Lambda$ *are algebraic ramification points. For every* $R > 0$ *there exist a positive integer* $N$ *such that for* $n > N$ *the* $\mu_n$ *are single-valued analytic functions on the set* $\{a : |a| < R\}$ *with disjoint graphs.*

Here are some general properties of implicit functions $\lambda(a)$ defined by equations of the form (7) with arbitrary entire function $F$. Let $g_0$ be an analytic germ of such function at some point $a_0$, and $\gamma_0 : [0,1] \to \mathbf{C}^{d-1}$ a curve in the $a$-space beginning at $a_0$. Then for every $\epsilon > 0$ there is a curve $\gamma$ beginning at $a_0$, satisfying $|\gamma(t) - \gamma_0(t)| < \epsilon,\ t \in [0,1]$, and such that an analytic continuation of $g_0$ along $\gamma$ is possible. In other words, "the set of singularities" of $\Lambda$ is totally discontinuous. This is called the *Iversen property*, and it was proved by Julia [21], see also [35]. However, the "set of singularities" of $\Lambda$ in general can have non-isolated points, as can be shown by examples, [15].

Theorem 2 is in fact an easy consequence of the following known result.

**Theorem A.** *For every* $R > 0$ *there exists a positive integer* $N$ *such that for all* $a$ *in the ball* $|a| \leq R$ *we have the strict inequalities* $|\mu_{n+1}(a)| > |\mu_n(a)|$ *for all* $n \geq N$.

For a complete proof of this result we refer to Shin [28, Thm 1.7] who used his earlier paper [29] and the results of Sibuya [31]. Theorem A is derived from the asymptotic expansion for the eigenvalues $\mu_n$ in powers of $n$ which is uniform with respect to $a$ for $|a| \leq R$ [28, Theorem 1.2]. Similar asymptotic formula for the eigenvalue problem (5), (6) is also given in [17, Ch. III, §6] where it is derived with a different method.

To deduce Theorem 2 from Theorem A, we also need the Weierstrass Preparation Theorem [6, 19]:

**Theorem B.** *Let* $Z \subset \mathbf{C}^m$ *be the set of solutions of the equation (7), and* $(a_0, \lambda_0) \in Z$*. Suppose that* $F(a_0, \lambda) \not\equiv 0$*. Then there is a neighborhood* $V$ *of* $(a_0, \lambda_0)$ *such that in* $V$ *we have*

$$F(a, \lambda) = \left((\lambda - \lambda_0)^k + F_{k-1}(a)(\lambda - \lambda_0)^{k-1} + \ldots + F_0(a)\right) G(a, \lambda),$$

*where* $F_j$ *and* $G$ *are analytic functions in* $V$*, and* $G(a_0, \lambda_0) \neq 0$*, and* $F_j(a_0) = 0$ *for* $0 \leq j \leq k-1$.

So for each $a$ close to $a_0$ the equation $F(a,\lambda)=0$ with respect to $\lambda$ has $k$ roots close to $\lambda_0$, and these roots tend to $\lambda_0$ as $a\to a_0$.

*Proof of Theorem 2.* Application of Theorem A shows that for every $R>0$ there exists $N$ such that for $n>N$ the functions $\mu_n$ have disjoint graphs over $\{a:|a|<R\}$. Application of Theorem B to the solutions $\mu_n(a)$ of the equation $F(a,\mu_n(a))=0$ with $n>N$ shows that $k=1$ for all points $(a,\mu_n(a_0))$, and then the implicit function theorem implies that the $\mu_n$ are analytic for $n>N$. This proves the second part of Theorem 2. To prove the first part, consider a curve $\gamma:[0,1]\to\{a:|a|\leq R\}\subset\mathbf{C}^{d-1}$ such that $\Lambda$ has an analytic continuation $g_t$, $0\leq t<1$. Here $g_t$ is an analytic germ of $\Lambda$ at the point $\gamma(t)$. Suppose that $g_0(\gamma(0))=\mu_j(\gamma(0))$. By Theorem A there exists $N>j$ such that $|g_t(\gamma(t))|<|\mu_N(\gamma(t))|$ for all $t\in[0,1)$. As $\mu_N(a)$ is bounded for $|a|\leq R$, we conclude that $g_t(\gamma(t))$ is a bounded function on $[0,1)$, and there exists a sequence $t_k\to 1$ such that $g_{t_k}(\gamma(t_k))$ has a finite limit $\lambda_1$. Application of the Weierstrass Preparation theorem to the point $(\gamma(1),\lambda_1)$ shows that in fact $g_t(\gamma(t))\to\lambda_1$ as $t\to 1$, and $g_t$ either has an analytic continuation to the point $\gamma(1)$ along $\gamma$ or $\gamma(1)$ is a ramification point of some order $k$. This completes the proof.

An alternative proof can be given by using perturbation theory of linear operators instead of the Weierstrass Preparation theorem as it is done in [32]. However we notice that an analog of Theorem 2 does not hold for general linear operators analytically dependent of parameters, as examples in [22, p. 371-372] show. The crucial property of our operators (5), (6) is expressed by Theorem A.

Theorem 2 implies statements b) and c) of Theorem 1.

In the rest of the paper we prove statement a). We briefly describe the idea of the proof. Equation (7) which we now write as

$$F(\alpha,\lambda)=0, \tag{8}$$

defines an analytic set $Z\subset\mathbf{C}^2$ which consists of all pairs $(\alpha,\lambda)$ for which the problem (4) has a solution. We are going to show that this set $Z$ consists of exactly two irreducible components, which are also its connected components. To do this we introduce a special parametrization of the set $Z$ by a (not connected) Riemann surface $G$. As this parametrization $\Phi:G\to Z$ comes from the work of Nevanlinna [26], we call it the *Nevanlinna parametrization.*

To study the Riemann surface $G$ we introduce a function $W:G\to\overline{\mathbf{C}}$, which has the property that it is unramified over $\overline{\mathbf{C}}\backslash\{0,1,-1,\infty\}$. More precisely, this means that

$$W : G\backslash W^{-1}(\{0,1,-1,\infty\}) \to \overline{\mathbf{C}}\backslash\{0,1,-1,\infty\}$$

is a covering map. Then we study the monodromy action on the generic fiber of this map $W$. Using a description of $G$ and $W$ which goes back to Nevanlinna, we label the elements of the fiber by certain combinatorial objects (cell decompositions of the plane) and explicitly describe the monodromy action on this set of cell decompositions. Our explicit description shows that there are exactly two equivalence classes of this action, thus the Riemann surface $G$ consists of exactly two components.

In fact we not only prove that $G$ consists of two components but give in some sense a global topological description of the surface $G$, and thus of the set $Z$.

The plan of the paper is the following. In Sections 2 we collect all necessary preliminaries and construct $G$, $\Phi$ and $W$. In Sections 3 and 4 we discuss the cell decompositions of the plane needed in the study of the monodromy of the map $W : G \to \overline{\mathbf{C}}$. In Section 5 we compute this monodromy and complete the proof of statement a) in Theorem 1. In section 6 we briefly mention several other one-parametric families of linear differential operators with polynomial potentials which can be treated with the same method.

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## 2. Preliminaries.

Some parts of our construction apply to the general problem (5), (6) so we explain them for this general case. We include more detail than it is strictly necessary for our purposes because the papers [26] and [25] are less known nowadays than they deserve.

First we recall some properties of solutions of the differential equation (5). The proofs of all these properties can be found in Sibuya's book [31]. Every solution of this differential equation is an entire function of order $(d+2)/2$, where $d$ is the degree of $P$. To avoid trivial exceptional cases, we always assume that $d > 0$. We set $q = d + 2$ and divide the plane into $q$ disjoint open sectors

$$S_j = \{z : |\arg z - 2\pi j/q| < \pi/q\}, \quad j = 0, \ldots, q-1.$$

In what follows we will always understand the subscript $j$ as a residue modulo

$q$, so that, for example, $S_q = S_0$ etc. We call $S_j$ the *Stokes sectors* of the equation (5).

1. For each solution $y \neq 0$ of the equation (5) and each sector $S_j$ we have either $y(z) \to 0$ or $y(z) \to \infty$ as $z \to \infty$ along each ray from the origin in $S_j$. We say that $y$ is *subdominant* in $S_j$ in the first case and *dominant* in $S_j$ the second case.

2. Of any two linearly independent solutions of (5), at most one can be subdominant in a given Stokes sector.

Let $y_1$ and $y_2$ be two linearly independent solutions, and consider their ratio $f = y_2/y_1$. Then $f$ is a meromorphic function of order $q/2$. (The order of a meromorphic function $f$ can be defined as the minimal number $\rho$ such that $f$ is a ratio of two entire functions of order at most $\rho$.)

3. For each $S_j$, we have $f(z) \to w_j \in \overline{\mathbf{C}}$ as $z \to \infty$ along any ray in $S_j$ starting at the origin.

4. $w_j \neq w_{j+1}$ for all $j \bmod q$.

5. $w_j \in \{0, \infty\}$ if and only if one of the solutions $y_1, y_2$ is dominant and another is subdominant in $S_j$.

A curve $\gamma : [0, 1) \to \mathbf{C}$ is called an *asymptotic curve* of a meromorphic function $f$ if $\gamma(t) \to \infty$ as $t \to 1$, and $f(\gamma(t))$ has a limit, finite or infinite, as $t \to 1$. This limit is called an *asymptotic value* of $f$. A classical theorem of Hurwitz says that the singularities of the inverse function $f^{-1}$ are exactly the critical values and the asymptotic values of $f$.

Returning to the function $f = y_2/y_1$, where $y_1$ and $y_2$ are linearly independent solutions of equation (5), we notice that $f$ does not have critical points. Indeed, all poles of $f$ are simple because $y_1$ can have only simple zeros, and $f'(z) \neq 0$ because the Wronskian determinant of $y_1, y_2$ is constant. Thus $f$ has no critical values, and the only singularities of $f^{-1}$ are the asymptotic values of $f$.

Next we describe these asymptotic values and associated asymptotic curves. By property 3 above, for each sector $S_j$, every ray from the origin in $S_j$ is an asymptotic curve. Thus all $w_j$ are asymptotic values. Function $f$ has no other asymptotic values except the $w_j$.

By Hurwitz theorem we conclude that the only singularities of $f^{-1}$ lie over the points $w_j$. More precisely,

$$f : \mathbf{C}\backslash f^{-1}(\{w_0, \dots, w_{q-1}\}) \to \overline{\mathbf{C}}\backslash\{w_0, \dots, w_{q-1}\} \tag{9}$$

is an unramified covering.

Let $D_j$ be discs centered at $w_j$ and having disjoint closures. Each component $B$ of the preimage $f^{-1}(D_j)$ is either a topological disc in the plane which is mapped by $f$ onto $D_j$ homeomorphically, or an unbounded domain such that $f : B \to D_j\backslash\{w_j\}$ is a universal covering. Such unbounded domains are called *tracts* over $w_j$.

6. The tracts are in bijective correspondence with the sectors $S_j$. More precisely, for each $j$ there exists a unique tract $B_j$ which contains each ray from the origin in $S_j$, except a bounded subset of this ray, and the total number of tracts is $q$.

The *Schwarzian derivative* of a function $f$ is

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The main fact about $S_f$ that we need is its relation with equation (5):

7. A ratio $f = y_2/y_1$ of two linearly independent solutions of (5) satisfies the differential equation

$$S_f = -2(P - \lambda), \tag{10}$$

and conversely, every non-zero solution of the differential equation (10) is a ratio of two linearly independent solutions of (5).

Let $Z_d \subset \mathbf{C}^d$ be the set of all pairs $(a, \lambda)$ such that $\lambda$ is an eigenvalue of the problem (5), (6). Consider the class $G_d$ of meromorphic functions with the following two properties:

$$-\frac{1}{2}S_f \quad \text{is a monic polynomial of degree } d \tag{11}$$

and

$$f(z) \to 0, \quad z \in \mathbf{R}, \quad z \to \pm\infty. \tag{12}$$

The set $G_d$ is equipped with the usual topology of uniform convergence on compact subsets of $\mathbf{C}$ with respect to the spherical metric in the target.

Now we define a map $\Phi : G_d \to \mathbf{C}^d$ by $\Phi(f) = (a_1, \ldots, a_{d-1}, \lambda)$, where $-\lambda, a_1, \ldots, a_{d-1}$ are the coefficients of the polynomial $-(1/2)S_f$. This map is evidently continuous.

**Proposition 1.** *The map $\Phi$ sends $G_d$ to $Z_d$ surjectively.*

*Proof.* First we prove that the image of $\Phi$ is contained in $Z_d$. Let $f$ be an element of $G_d$. By property 7 above, $f = y/y_1$, a ratio of two linearly independent solutions of (5). By (12) and property 5 above, $y$ should be subdominant in $S_0$ and $S_{q/2}$. So $y$ satisfies the boundary condition (6) and thus $\Phi(f) = (a, \lambda)$ is an element of $Z_d$.

Now we prove that $\Phi$ maps $G_d$ to $Z_d$ surjectively. Let $(a, \lambda) \in Z_d$ and let $y$ be the corresponding eigenfunction. Let $y_1$ be any solution of the equation (5) which is linearly independent of $y$. Then $f = y/y_1$ satisfies the differential equation (10), thus (11) holds. Now in view of the boundary condition (6), $y$ is subdominant in $S_0$ and $S_{q/2}$, so by properties 1 and 2 above $y_1$ must be dominant in $S_0$ and $S_{q/2}$. So $f = y/y_1$ satisfies (12). Thus $f \in G_d$, and (10) gives $(a, \lambda) = \Phi(f)$.

**Proposition 2.** *A meromorphic function $g$ satisfies $g(z) = f(cz)$ for some $f \in G_d$ and $c \in \mathbf{C}^*$ if and only if it has the following three properties:*

(i) *$g$ has no critical points,*

(ii) *$g$ has $q = d + 2$ tracts,*

(iii) *There is a tract $B_0$ such that if the tracts are ordered counterclockwise as $B_0, B_1, \ldots, B_{q-1}$, and if $w_j$ are the asymptotic values of $g$ in $B_j$ then $w_0 = w_{q/2} = 0$.*

The sufficiency of these conditions is a deep result of R. Nevanlinna [26]. The proof becomes much simpler if one adds the condition

(iv) *$g$ is a meromorphic function of finite order.*

This additional condition will be easy to verify in our setting and we sketch a simpler proof of Proposition 2, using the condition (iv). This proof is based on F. Nevanlinna's work [25].

*Proof of Proposition 2 with condition* (iv). The necessity of conditions (i)-(iv) has already been established.

Now we prove sufficiency. Condition (i) implies that $S_g$ is an entire function (in general, Schwarzian derivative of a meromorphic function has poles exactly at its critical points). Condition (iv) combined with the Lemma on the logarithmic derivative [27, 18] gives a growth estimate of $S_g$ which implies that $S_g$ is a polynomial. Now by property 7 above, $g = y/y_1$, where $y, y_1$ are two linearly independent solutions of the differential equation

$$y'' + \frac{1}{2} S_g y = 0.$$

Property 6 above shows that $g$ has $\deg S_g + 2$ tracts, so we conclude from (ii) that $\deg S_g = d$. Now we can find $c \in \mathbf{C}^*$ so that for $f(z) = g(z/c)$ the polynomial $(-1/2)S_f$ is monic. Such $c$ is defined up to multiplication by a $q$-th root of unity. Using (iii), we choose this root of unity in such a way that (12) is satisfied.

Now we define a map $W : G_d \to \overline{\mathbf{C}}^q$, $W(f) = (w_0, \ldots, w_{q-1})$, whose image is evidently contained in the subspace $H$ of codimension 2 given by the equations $w_0 = w_{q/2} = 0$. This map is known to be a local homeomorphism into $H$ [2], and its image can be described precisely using a result of R. Nevanlinna [26]. We do not use these results in our paper. We only remark that the local homeomorphism $W$ permits to define a structure of a complex analytic manifold of dimension $d$ on $G_d$, so that $W$ becomes holomorphic. The map $\Phi$ we introduced earlier is also holomorphic with respect to this analytic structure.

**Proposition 3.** *Let $f_0$ be an element of $G_d$, and $\gamma : [0,1] \to H$,*

$$\gamma(t) = (w_0(t), \ldots, w_{q-1}(t)), \quad w_0 \equiv w_{q/2} \equiv 0,$$

*be a path with the properties $\gamma(0) = W(f_0)$, and*

$$w_j(0) \neq w_k(0) \quad \Longleftrightarrow \quad w_j(t) \neq w_k(t)$$

*for all $j \neq k$ and all $t \in [0,1]$. Then there is a lift of the path $\gamma$ to $G_d$, that is a continuous family $f_t \in G_d$, $t \in [0,1]$ such that $W(f_t) = \gamma(t)$.*

*Proof.* There is a continuous family of diffeomorphisms $\psi_t : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, $\psi_0 = \mathrm{id}$, such that $\psi_t(w_j(0)) = w_j(t)$, $0 \leq j \leq q-1$. These diffeomorphisms are quasiconformal [1]. Then the Fundamental existence theorem for quasiconformal maps [1, Chap. V] implies the existence of a continuous family of quasiconformal maps $\phi_t$, $\phi_0 = \mathrm{id}$, such that $g_t = \psi_t \circ f_0 \circ \phi_t$ are meromorphic functions. These meromorphic functions evidently have no critical points, because $g_0$ does not. They have the same number of tracts as $g_0$ and their tracts satisfy the condition (iii) of Proposition 2. Thus all conditions of Proposition 2 are satisfied. We can also check the additional condition (iv): it follows from the general property of quasiconformal mappings $|\phi_t(z)| \leq |z|^C, |z| > r_0$, where $C$ is a constant. Thus by Proposition 2, $g_t = f_t(c_t z)$, $f_t \in G_d$, where the constants $c_t$ are determined from the condition that the polynomials $-(1/2)S_{f_t}$ are monic. Evidently the correspondence $t \mapsto c_t$ is continuous, and we have $W(f_t) = \gamma(t)$ by construction.

*Centrally symmetric case.*

Suppose now that the polynomial $P$ in (5) is even. We write it as $P(a, z) = z^d + a_{d-2}z^{d-2} + \ldots + a_2 z^2$ and consider the set $Z_d^e$ of all pairs $(a, \lambda) \in \mathbf{C}^{d/2}$ such that $\lambda$ is an eigenvalue of the problem (5), (6). Then each eigenfunction $y$ of the problem (5), (6) is either even or odd. Indeed, $y(-z)$ is also an eigenfunction with the same eigenvalue, so $y(z) = cy(-z)$ because the eigenspace is one-dimensional. Putting $z = 0$ we obtain that either $c = 1$ (so the eigenfunction is even) or $y(0) = 0$. In the latter case, differentiate to obtain $y'(z) = -cy'(-z)$ and put $z = 0$ to conclude that $c = -1$, so the eigenfunction is odd.

Equation (5) with even $P$ always has even and odd solutions: to obtain an even solution we solve the Cauchy problem with the initial conditions $y_1(0) = 1, y_1'(0) = 0$; to obtain an odd solution we use the initial conditions $y_1(0) = 0, y_1'(0) = 1$.

Let $y$ be an eigenfunction, and $y_1$ a solution of (5) of the *opposite parity* to $y$. Then $y$ and $y_1$ are linearly independent, and the ratio $f = y/y_1$ is odd. Let $G_d^o$ be the set of all odd functions in $G_d$. Then $\Phi$ maps $G_d^o$ to $Z_d^e$ because the Schwarzian derivative of an odd function is even. Thus we have a centrally symmetric version of Proposition 1: the map

$$\Phi : G_d^o \to Z_d^e \quad \text{is well defined and surjective.}$$

Similarly, Proposition 2 has a centrally symmetric analog: *for an odd meromorphic function $g$ to be of the form $f(cz)$, where $f \in G_d^o$, it is necessary and sufficient that conditions* (i)-(iii) *(or* (i)-(iv)*) be satisfied.* Finally, Proposition 3 has a centrally symmetric analog:

**Proposition 3′.** *Let $f_0$ be an element of $G_d^o$, and $\gamma : [0,1] \to H$,*

$$\gamma(t) = (w_0(t), \ldots, w_q(t)), \quad w_0 \equiv w_{q/2} \equiv 0,$$

*be a path with the properties $\gamma(0) = W(f_0)$, $w_j(t) = -w_{j+q/2}(t)$ and*

$$w_j(0) \neq w_k(0) \quad \iff \quad w_j(t) \neq w_k(t)$$

*for all $j \neq k \bmod q$ and all $t \in [0, 1]$. Then there is a lift of the path $\gamma$ to $G_d^o$, that is a continuous family $f_t \in G_d^o$, $t \in [0, 1]$ such that $W(f_t) = \gamma(t)$.*

The proof is the same as that of the original Proposition 3: one can choose all homeomorphisms $\psi_t$ and $\phi_t$ to be odd, then $g_t = \psi_t \circ f_0 \circ \phi_t$ will be odd.

Case a) of Theorem 1 which we are proving corresponds to the even potential with $d = 4$. To prove a) we only need to show that $G_4^o$ consists of two components: one containing the functions with $f(0) = 0$ and another containing the functions with $f(0) = \infty$.

Notice that $\mathbf{C}^*$ acts on $G_d$ and on $G_d^o$ by the rule $f \mapsto cf$, and that the map $\Phi$ is invariant with respect to this action. Introducing the equivalence relation $f \sim cg$, $c \in \mathbf{C}^*$ on $G_d^o$ we obtain a factor-map $\tilde{\Phi}$ which maps the equivalence classes to $Z_d$.

**Proposition 4.** *The map $\tilde{\Phi} : G_d^o / \sim \to Z_d^e$ is a homeomorphism.*

*Proof.* We only have to show that it is injective, that is that any two non-zero odd solutions $f_1$ and $f_2$ of the Schwarz differential equation $S_f = -2P$, which tend to zero as $z \to \pm\infty$ on the real line, are proportional. All non-zero solutions of a Schwarz differential equation are related by fractional-linear transformations. So we have $f_1 = T \circ f_2$, where $T$ is a fractional-linear transformation. Changing $z$ to $-z$ we conclude that $T$ is odd. Every odd fractional-linear transformation has the form $cz$ or $c/z$. The latter case is excluded by the condition that $f_1(z)$ and $f_2(z)$ both tend to zero as $z \to \infty$ on the real line. Thus $f_1 = cf_2$.

In the next section we study the map $W : G_d \to \overline{\mathbf{C}}^q$, and in particular, the monodromy of this map. For this we need a description of the general fiber of this map by certain cell decompositions of the plane. Notice that our map $W$ commutes with multiplication by constants $c \in \mathbf{C}^*$.

## 3. Some cell decompositions of the plane.

By a *cell decomposition* of a surface $X$ we understand its representation as a locally finite union of disjoint subsets called *cells*. The cells can be of dimension 0 (points or vertices), 1 (edges) or 2 (faces). The edges and faces are homeomorphic images of an open interval or of an open disc, respectively, and they satisfy the following condition: the boundary (in $X$) of each cell is a locally finite union of cells of smaller dimension of this decomposition. We do not assume that the homeomorphisms of the open discs defining faces have extensions to the closed discs.

Let $w = f(z)$ be a meromorphic function without critical points and with finitely many asymptotic values. Consider a fixed cell decomposition $\Psi_0$ of the sphere $\overline{\mathbf{C}}_w$ such that all asymptotic values are contained in the faces and each face contains one asymptotic value. Then the preimage $\Psi_f = f^{-1}(\Psi_0)$ is a cell decomposition of the plane $\mathbf{C}_z$ with connected 1-skeleton (The 1-skeleton of a cell decomposition is the union of edges and vertices.) That the 1-skeleton is connected is seen from (9), which is a covering, and from the fact that every path in $\overline{\mathbf{C}}\backslash\{w_0, \ldots, w_{q-1}\}$ can be deformed to a path in the 1-skeleton of $\Psi_0$. The closures of the edges of $\Psi_f$ are mapped by $f$ onto the closures of the edges of $\Psi_0$ homeomorphically. Each face $B$ of $\Psi_f$ is mapped by $f$ onto a face $D$ of $\Psi_0$ either homeomorphically or as a universal covering over $D\backslash\{w\}$ where $w$ is the asymptotic value in $D$. In the former case the face $B$ of $\Psi_f$ is bounded, in the latter case it is unbounded.

We label the faces of $\Psi_f$ by the names of their image faces under $f$. Labeled cell decompositions of the plane $\mathbf{C}_z$ are considered up to equivalences, orientation-preserving homeomorphisms of the plane preserving the labels. If $f$ is an odd function, it is reasonable to choose $\Psi_0$ to be invariant under the map $w \mapsto -w$. Then $\Psi_f$ will be also invariant under $z \mapsto -z$. For such cell decompositions of $\mathbf{C}_z$, the natural equivalence relation is that they are mapped one onto another by an odd orientation-preserving homeomorphism of $\mathbf{C}_z$ respecting the face labels. We call two such cell decompositions symmetrically equivalent. For a given set of asymptotic values and a given $\Psi_0$, the labeled cell decomposition $\Psi_f$ almost completely determines $f$. Namely, we have the following

**Proposition 5.** *Let $f_1$ and $f_2$ be two meromorphic functions without critical points and with the same finite set of asymptotic values. Fix a cell decomposition $\Psi_0$ of $\overline{\mathbf{C}}_w$ such that the asymptotic values are contained in faces of $\Psi_0$ and each face contains one asymptotic value. If $\Psi_i = f_i^{-1}(\Psi_0)$ are equivalent cell decompositions of $\mathbf{C}_z$ then $f_1(z) = f_2(cz + b)$, $c \neq 0$.*

*If $f_i$ are odd, $\Psi_0$ is centrally symmetric, and the $\Psi_i$ are symmetrically equivalent then $b = 0$.*

*Proof.* Let $\psi$ be the orientation-preserving homeomorphism of the plane

$\mathbf{C}_z$ that maps $\Psi_1$ onto $\Psi_2$ preserving the face labels. We are going to define another homeomorphism $\psi'$ with the same properties, and in addition,

$$f_1 = f_2 \circ \psi'. \tag{13}$$

Let $B_1$ and $B_2 = \psi(B_1)$ be two faces such that $f_i$ map $B_i$ onto a face $B_0$ of $\Psi_0$. If one of the $B_1, B_2$ is bounded then another is also bounded and the maps $f_i : B_i \to B_0$ are homeomorphisms. So there exists a unique homeomorphism $\psi' : B_1 \to B_2$ such that (13) holds. If both $B_1$ and $B_2$ are unbounded, then $f_i : B_i \to B_0 \backslash \{w_0\}$ are universal coverings, and there are infinitely many homeomorphisms $B_1 \to B_2$ that satisfy (13). To choose one, we first notice that every homeomorphism $B_1 \to B_2$ with property (13) has a continuous extension to the boundary $\partial B_1$ and sends boundary edges of $B_1$ to boundary edges of $B_2$. We choose $\psi'$ in $B_1$ so that it maps the boundary edges in the same way as $\psi$. This is possible to do as $\psi$ preserves orientation. Now $\psi'$ is defined on all faces of $\Psi_1$, and it is easy to check that the boundary extensions from different faces to edges match. Thus $\psi'$ is a homeomorphism of the plane satisfying (13), and (13) implies that it is conformal. So $\psi'(z) = az + b$. It is easy to check that in the centrally symmetric case the above construction gives an odd homeomorphism $\psi'$.

Now we consider a special class of cell decompositions $\Psi_0$ which is convenient for our purposes.[2]

Let $f : \mathbf{C}_z \to \overline{\mathbf{C}}_w$ be a meromorphic function of order $q/2$ without critical points and with the asymptotic values $w_0, \ldots, w_{q-1}$, ordered according to the cyclic order of their Stokes sectors. We assume that the set $J = \{j : w_j \neq 0\}$ is a fixed subset of $\{0, \ldots, q-1\}$, and that all nonzero asymptotic values of $f$ are finite and distinct. Let $J = \{j_1, \ldots, j_k\}$ with $j_1 < \ldots < j_k$, and let $c_\nu = w_{j_\nu}$. Then the cyclic order of the Stokes sectors with nonzero asymptotic values $c_1, \ldots, c_k$ agrees with that of $\{1, \ldots, k\} \bmod k$. We assume $3 \leq k < q$, so $f$ has at least three distinct non-zero asymptotic values.

We define a cell decomposition $\Psi_0$ of $\overline{\mathbf{C}}_w$ with a single vertex at $\infty$ as follows (see Fig. 1).

[2]The usual choice of $\Psi_0$ as in [13, 18, 26, 27] leads to the cell decompositions of the plane which are called line complexes. We prefer a different choice of $\Psi_0$, as in [16], which is better compatible with the symmetries of our problem.

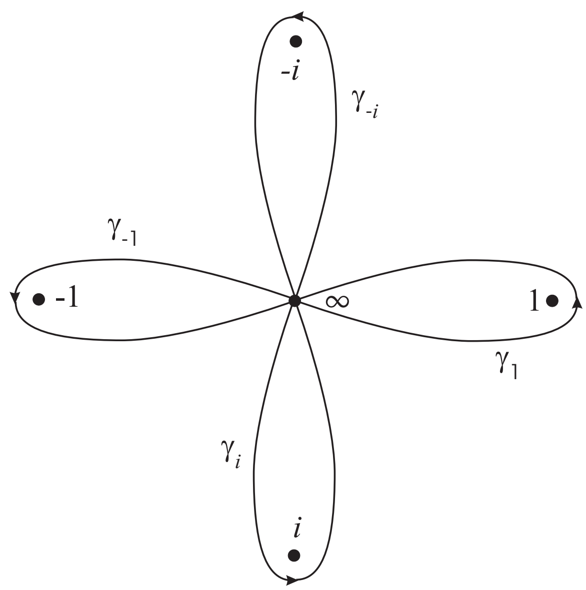


Fig. 1. Cell decomposition $\Psi_0$ for asymptotic values $0, \pm 1, \pm i$.

In the complex plane $\mathbf{C}^\bullet_w = \overline{\mathbf{C}}_w \backslash \{0\}$, fix a system of directed loops $\gamma_{c_1}, \ldots, \gamma_{c_k}$, starting and ending at $\infty$, intersecting only at their endpoints, and such that each loop $\gamma_{c_\nu}$ is an oriented boundary of an open domain $D_{c_\nu}$ containing $c_\nu$ and not containing other asymptotic values of $f$. Let $D_0$ be the connected component of 0 in $\overline{\mathbf{C}}_w \setminus (\gamma_{c_1} \cup \ldots \cup \gamma_{c_k})$. The domains $D_0$, $D_{c_1}, \ldots, D_{c_k}$ are the faces of $\Psi_0$, and the open loops $\dot{\gamma}_\nu = \gamma_\nu \setminus \{\infty\}$ are its edges.

There is a natural cyclic order $\nu_1 \prec \ldots \prec \nu_k \prec \nu_1$ of the loops $\gamma_{c_\nu}$, and of the corresponding domains $D_{c_\nu}$. It is defined by the order in which the domains $D_{c_\nu}$ cross the oriented boundary of a small disk in $\mathbf{C}^\bullet_w$ centered at $\infty$. Alternatively, it is opposite to the order in which the loops $\gamma_{c_\nu}$ appear in the boundary of $D_0$.

So we introduced two cyclic orders on $c_1, \ldots, c_k$, the first one coming from the cyclic order of the Stokes sectors, and the second one from the cyclic order of the loops in the cell decomposition $\Psi_0$. These two cyclic orders are in general different.

Let $\Psi_f = f^{-1}(\Psi_0)$ be the corresponding cell decomposition of $\mathbf{C}_z$. The vertices of $\Psi_f$ are the poles of $f$. The faces of $\Psi_f$ are labeled with 0, $c_1, \ldots, c_k$ and edges with $c_1, \ldots, c_k$. The edges of $\Psi_f$ are directed, being preimages of directed loops $\gamma_{c_\nu}$. Since $f$ has no critical points, the restriction of $f$ to any face of $\Psi_f$ is either a homeomorphism or a universal covering over the image face of $\Psi_0$ minus the asymptotic value in this image face. Accordingly, $\Psi_f$ has the following properties:

**(1)** Unbounded faces of $\Psi_f$ are in one-to-one correspondence with the Stokes sectors of $f$. This follows from property 6 in Section 2. For each $\nu = 1, \ldots, k$ there is exactly one unbounded face $B_{c_\nu}$ labeled with $c_\nu$. The faces $B_{c_1}, \ldots, B_{c_k}$ have the cyclic order
$\{1, \ldots, k\} \bmod k$ at infinity.

**(2)** Edges of $\Psi_f$ may be either *links* (having two distinct vertices) or *loops* (having both ends at the same vertex). Each edge labeled by $c_\nu$ separates two faces, labeled with $c_\nu$ and 0, respectively. Its orientation agrees with that of the boundary of its adjacent face labeled with $c_\nu$. If it is a link, it is adjacent to the unbounded face $B_{c_\nu}$.

**(3)** Each bounded face of $\Psi_f$ labeled with $c_\nu$ has as its boundary a loop labeled by $c_\nu$. The unbounded face $B_{c_\nu}$ has as its boundary an infinite chain of links labeled with $c_\nu$, and no loops. A bounded face labeled with 0 has as its boundary $k$ edges labeled with $c_\nu$, (oriented opposite to their natural orientation) in the cyclic order opposite to the order $\nu_1, \ldots, \nu_k$ of the loops $\gamma_{c_\nu}$. An unbounded face labeled with 0 has as its boundary an infinitely repeated sequence of $k$ edges labeled with $c_\nu$, in the cyclic order opposite to $\nu_1, \ldots, \nu_k$.

**(4)** The cyclic order of the values $c_\nu$ labeling the *links* in the boundary of a bounded face labeled by 0 is the same as the cyclic order of the unbounded faces $B_{c_\nu}$ adjacent to these links, which agrees with the cyclic order $\{1, \ldots, k\} \bmod k$ of the Stokes sectors.

**(5)** Each vertex $v$ of $\Psi_f$ has degree $2k$, with the directed edges labeled with $c_{\nu_1}, \ldots, c_{\nu_k}$ consecutively exiting and entering $v$, where $\nu_1, \ldots, \nu_k$ is the cyclic order of the loops $\gamma_{c_\nu}$. This means that, as an edge labeled with $c_{\nu_1}$ exits $v$, the next in the cyclic order is (the same or another) edge labeled with $c_{\nu_1}$ entering $v$, then an edge labeled with $c_{\nu_2}$ exiting $v$, and so on, till an edge labeled with $c_{\nu_k}$ entering $v$, followed by the initial edge labeled with $c_{\nu_1}$ exiting $v$.

The one-skeleton of $\Psi_f$ is an infinite directed graph properly embedded in $\mathbf{C}_z$. It is connected.

Removing from the 1-skeleton of $\Psi_f$ all loops, we obtain a directed graph $\Gamma$. From property **(3)** of $\Psi_f$, all bounded components of the complement of $\Gamma$ are labeled by 0, and the unbounded components are in one-to-one correspondence with the Stokes sectors of $f$. Moreover, the unbounded components corresponding to the Stokes sectors with nonzero asymptotic values are exactly the faces $B_{c_\nu}$ of $\Psi_f$. Replacing two edges in the boundary of each two-gon of $\Gamma$ by one *undirected* edge inside the two-gon connecting its two vertices, and forgetting orientation of all remaining edges of $\Gamma$, we obtain a properly embedded graph $T$ without loops or multiple edges, with the components of its complement labeled with 0 and $c_\nu$. That $\Gamma$ has no multiple edges follows from property **(2)** of $\Psi_f$: each link in $\Psi_f$ belongs to the boundary of some unbounded face, thus there are at most two links in $\Phi_f$ between any pair of vertices. Each edge of $T$ separates two components with different labels. All bounded components are labeled with 0.

**Proposition 6.** *Suppose that the cyclic order $\nu_1, \ldots, \nu_k$ of the loops $\gamma_{c_\nu}$ in $\mathbf{C}_w^\bullet$ agrees with the cyclic order $\{1, \ldots, k\} \bmod k$ of the Stokes sectors with the asymptotic values $c_1, \ldots, c_k$ in $\mathbf{C}_z$. Then each bounded face of $\Psi_f$ labeled with 0 has exactly two vertices, and its boundary contains exactly two links. Each bounded component of the complement of $\Gamma$ is a two-gon, and $T$ is an embedded planar tree.*

*Proof.* First, a bounded face $C$ of $\Psi_f$ labeled with 0 must have at least two vertices. Otherwise, its boundary would consist of loops labeled with $c_\nu \neq 0$, which should be also boundaries of bounded faces labeled with $c_\nu$. This is impossible since the union of this face with the loops and the vertices would be a sphere, and could not be embedded in $\mathbf{C}_z$. Hence there are at least two links in the boundary of $C$.

From property **(3)** of $\Psi_f$, the cyclic order of the links labeled with $c_\nu$ in the boundary of $C$ should be opposite to the cyclic order of the loops $\gamma_{c_\nu}$. From property **(4)**, it should agree with the cyclic order $\{1, \ldots, k\} \bmod k$ of the faces $B_{c_\nu}$. If the loops $\gamma_{c_\nu}$ and the faces $B_{c_\nu}$ have the same cyclic order, this is impossible when the boundary of $C$ contains more than two links.

Since $\Gamma$ is obtained by removing loops from the one-skeleton of $\Psi_f$, its complement has no bounded components other than two-gons, hence the complement of $T$ has no bounded components, so $T$ is a forest (a union of disjoint trees).

To prove that it is connected, we notice that the 1-skeleton of $\Psi_f$ is connected, and the removal of loops and multiple edges from $\Psi_f$ does not affect this connectedness.

**Proposition 7.** *The embedded planar directed graph $\Gamma$ and the cell decomposition $\Psi_f$ are determined by the embedded planar graph $T$ uniquely up to* *an orientation-preserving homeomorphism of* $\mathbf{C}_z$ *preserving their common vertices.*

*Proof.* The components of the complement of $T$ labeled with 0 coincide with the components of the complement of $\Gamma$ labeled with 0. A unique unbounded component $C_\nu$ of the complement of $T$ labeled by $c_\nu$ contains the unbounded face $B_{c_\nu}$ of $\Psi_f$. Each of its boundary edges separates $C_\nu$ either from a component of the complement of $T$ labeled with 0 or from $C_\mu$ with $\mu \neq \nu$. In the first case, the edge belongs to $\Gamma$. We make it directed as part of the boundary of $C_\nu$ and label it with $c_\nu$. Otherwise, we connect the two vertices of the edge by a new edge labeled with $c_\nu$ inside $C_\nu$, directed according to the orientation of the boundary of $C_\nu$, so that all these new edges are disjoint (this can be done one edge at a time, in any order). As a result, each edge of $T$ separating two components $C_\nu$ and $C_\mu$ is included inside a two-gon. Let $\Gamma'$ be the embedded planar directed graph obtained by removing all such edges of $T$. We want to show that $\Gamma'$ can be obtained from $\Gamma$ by an orientation-preserving homeomorphism of $\mathbf{C}_z$ preserving all vertices and labels. First, two-gons of $\Gamma'$ have the same vertices and the same labeling of edges as the two-gons of $\Gamma$. Hence there exists an orientation-preserving homeomorphism between each two-gon of $\Gamma$ and the two-gon of $\Gamma'$ with the same vertices, preserving their common vertices. These homeomorphisms define a homeomorphism between the union of all two-gons of $\Gamma$ and the union of all two-gons of $\Gamma'$, preserving their common vertices and the labeling of their edges. It can be extended to unbounded components to obtain a homeomorphism of $\mathbf{C}_z$.

From property **(5)** of $\Psi_f$, the cyclic order of the components $B_{c_\nu}$ adjacent to a vertex $v$ of $\Gamma$ agrees with the cyclic order $\nu_1, \ldots, \nu_k$ of the loops $\gamma_{c_\nu}$. The cyclic order of the edges of $\Gamma$ exiting and entering $v$ is determined by the cyclic order of the unbounded components $B_{c_\nu}$ of its complement adjacent to $v$. Since this order agrees with the cyclic order of the faces $D_{c_\nu}$ of $\Psi_0$ (same as the order of the loops $\gamma_{c_\nu}$), one can add to $\Gamma$ non-intersecting loops, having both ends at $v$, labeled with the missing values $c_\nu$ inside the connected components of the complement of $\Gamma$ labeled by 0 adjacent to $v$, so that the cyclic order of the edges consecutively exiting and entering $v$ becomes $\nu_1, \ldots, \nu_k$. Labeling the interiors of these loops by the corresponding values $c_\nu$, we obtain a cell decomposition of $\mathbf{C}_z$ that can be obtained from $\Psi_f$ by an orientation-preserving homeomorphism of $\mathbf{C}_z$ preserving vertices and labels (this can be done first for the loops, in any order, then extended to components labeled with 0).

*Centrally symmetric case.*

Suppose now that the set of asymptotic values $w_j$ (and the corresponding set of non-zero values $c_\nu$) is centrally symmetric. In this case, $k$ is even and $c_{\nu+k/2} = -c_\nu$. We can choose the loops $\gamma_{c_\nu}$ centrally symmetric (e.g., by selecting $k/2$ loops about $c_\nu^2$ and taking square roots of them). If $f$ is odd, the cell decomposition $\Psi_f$ is centrally symmetric, and so are the graphs $\Gamma$ and $T$ (assuming the edges of $T$ inside centrally symmetric two-gons of the complement of $\Gamma$ are chosen centrally symmetric). The origin $0 \in \mathbf{C}_z$ is either a vertex of $\Psi_f$ when $f(0) = \infty$, or the center of a bounded face labeled with 0 when $f(0) = 0$. This bounded face may correspond either to a bounded face of the complement of $T$ (if its boundary has more than two links) or to the middle of an edge of $T$.

## 4. Case $q = 6$, $k = 4$: classification of trees.

Consider the centrally symmetric case $q = 6$, $k = 4$, with $w_0 = w_3 = 0$. We assume that the Stokes sector $S_0$ contains a ray of the positive real axis. We have $c_1 = w_1 = -c_3 = -w_4$ and $c_2 = w_2 = -c_4 = -w_5$ satisfying $\{c_1, c_2\} \cap \{0, \infty\} = \emptyset$ and $c_1 \neq \pm c_2$. Since multiplication of all asymptotic values by a nonzero constant corresponds to multiplication of $f$ by the same constant, we can assume $c_2 = 1$ and $c_1 = c \neq 0, \infty, \pm 1$.

Suppose that the cell decomposition $\Psi_0$ is centrally symmetric and satisfies conditions of Proposition 6, so that $T$ is a centrally symmetric tree. The two components of its complement labeled by 0 and opposite via central symmetry, cannot have a common boundary edge.

**Proposition 8.** *Consider centrally symmetric embedded trees in* $\mathbf{C}$ *with six ends, two opposite components of the complement labeled by zero, and such that these two components do not have a common edge. Any such tree is equivalent to either one of the trees shown in Fig. 2, or its complex conjugate. The integer parameters satisfy the following restrictions: for* $A_k :\ k \geq 0$, *for* $D_{k,l} :\ k \geq 0, l \geq 1$, *and for* $E_{k,l} :\ k \geq 1, l \geq 0$.

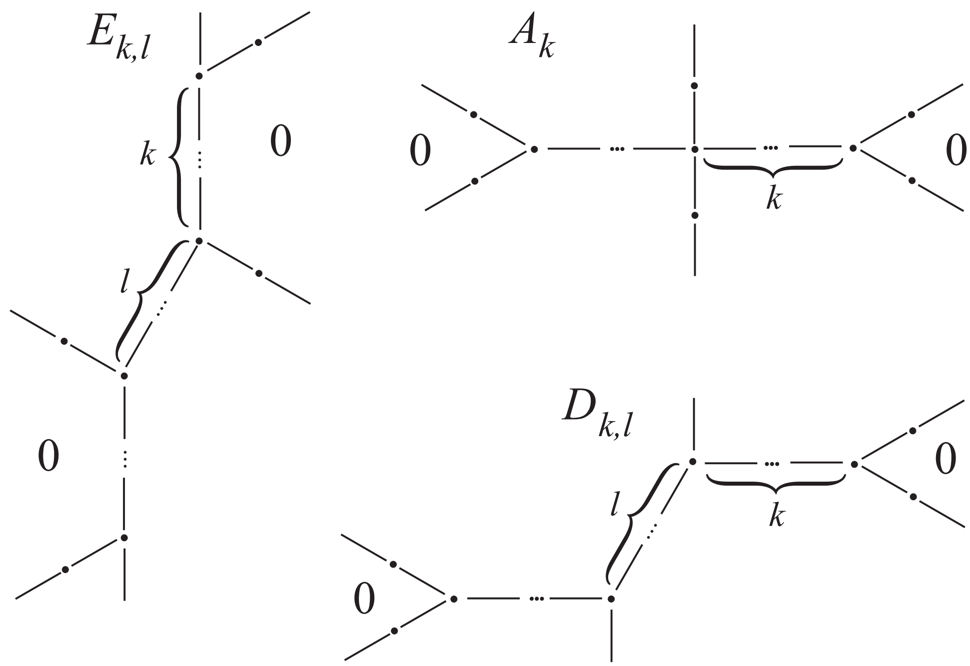


Fig. 2. Classification of trees.

*Proof.* Let $X$ be a graph in $\mathbf{C}_z$ with the vertices corresponding to components of the complement of $T$ and the edges connecting two of its vertices if the corresponding components have a common edge in $T$. Then $X$ is combinatorially a hexagon with some of the chords connecting its vertices, so that the chords do not intersect and the vertices 0 and 3 not connected. We can also assume that $X$ is centrally symmetric and that its vertex labeled with 0 is on the positive real axis. There are ten possible cases. Six of them are listed in Fig. 3. They correspond to embedded planar trees $A_k$, $D_{k,l}$, $E_{k,l}$ shown in Fig. 2. Here indices $k$ and $l$ denote the number of edges in a chain of edges and simple vertices of $T$ corresponding to a chord of $X$ (with $l$ corresponding to the chord passing through the origin). In particular, a tree with even $l$ has a vertex at the origin, while a tree with odd $l$ has the origin as the middle of its edge. The remaining four cases are obtained by complex conjugation.

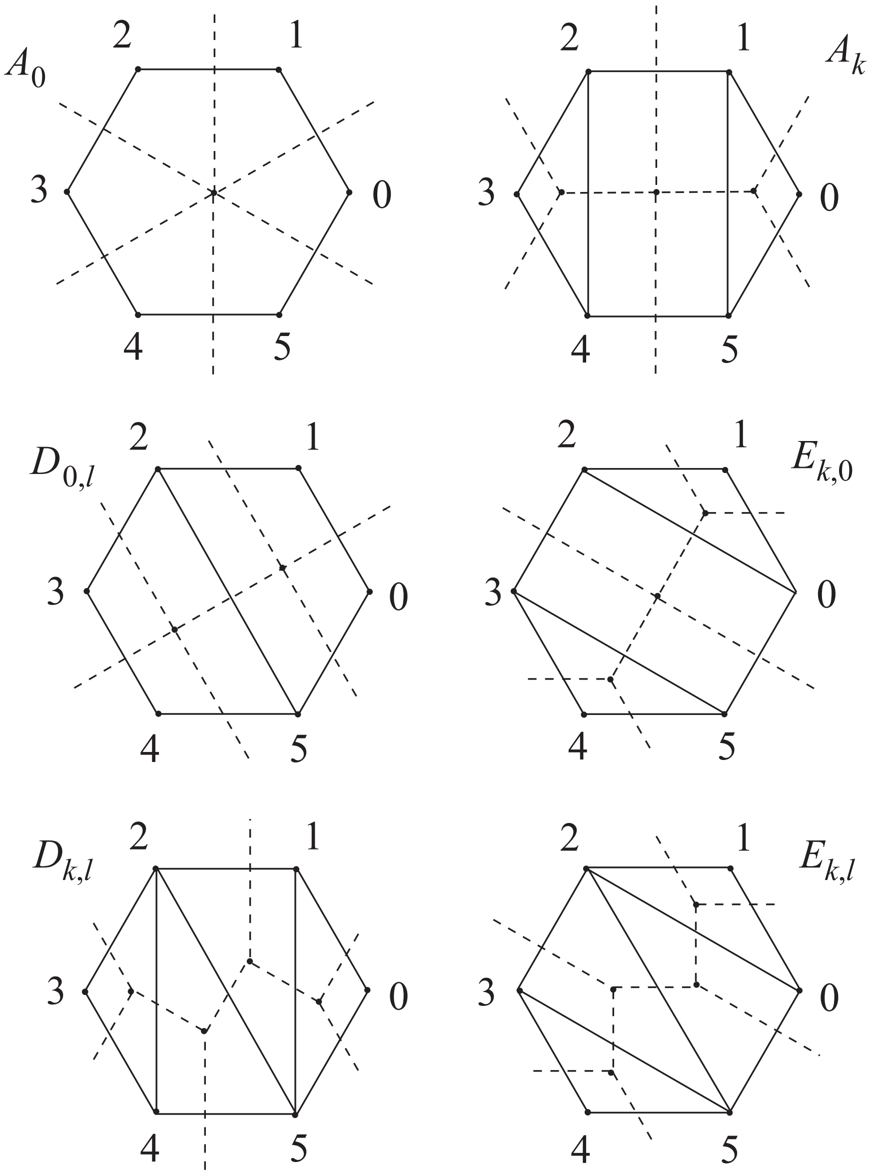


Fig. 3. To the proof of Proposition 8.

The trees $A_k$ are symmetric with respect to complex conjugation. The complex conjugates of the other trees are denoted by $\bar{D}_{k,l}$ and $\bar{E}_{k,l}$.

If the condition of Proposition 6 is not satisfied (i.e., the cyclic order of the loops in $\Psi_0$ is different from the cyclic order of the Stokes sectors) the graph $T$ may still be a tree when no three components of its complement labeled by nonzero values are adjacent to any of its vertices. This is the case for the trees $E_{k,l}$ and their complex conjugates $\bar{E}_{k,l}$. To distinguish them from the same trees with the correct cyclic order of face labels we denote them by $E'_{k,l}$ and $\bar{E}'_{k,l}$, respectively. Examples of non-tree graphs $T$ are shown in Fig. 4.

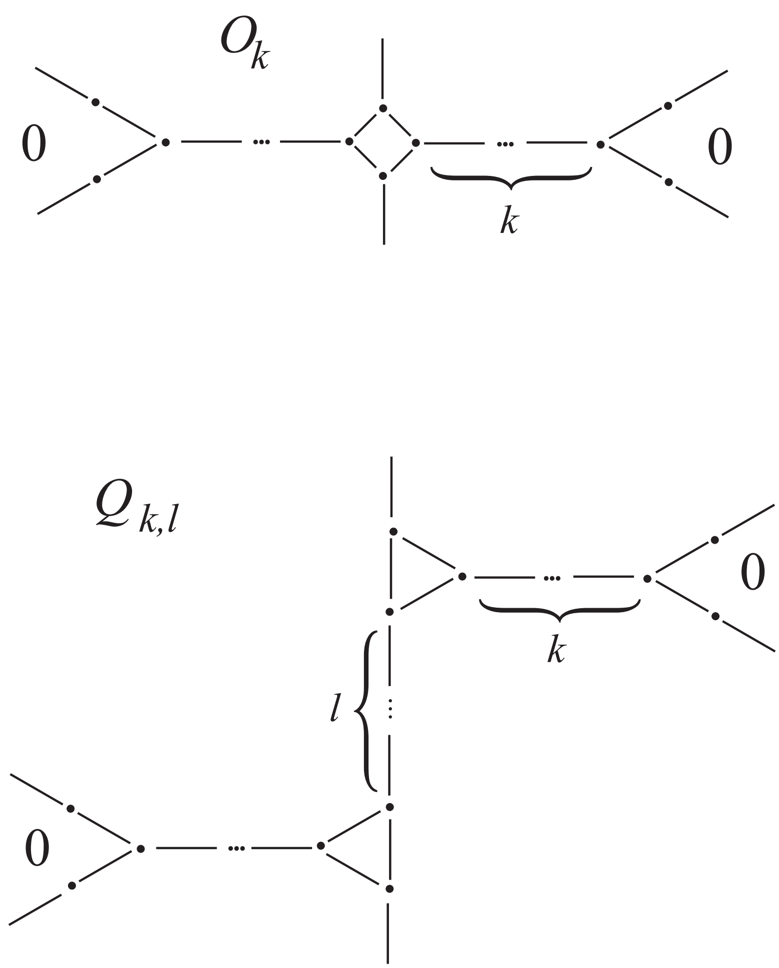


Fig. 4. Non-tree examples of graphs $T$.

These graphs are $O_k$ and $Q_{k,l}$, with $k, l \geq 0$, and the complex conjugates $\bar{Q}_{k,l}$ of $Q_{k,l}$. We will later show that these graphs $T$ can really occur. Moreover, these are all possible cases of non-tree graphs $T$ corresponding to cell decompositions $\Psi_f$ with the reverse cyclic order of non-zero labels, but we do not use this fact in the proof; it will rather come as a consequence of our arguments.

**5. Monodromy. Proof of Theorem 1 a).**

Let $\Xi$ be the space of all four-point sequences $\xi = \{c_1, \ldots, c_4\}$ in $\overline{\mathbf{C}}_w$ such that $c_3 = -c_1$, $c_4 = -c_2$, $c_\mu \neq 0, \infty, \pm c_\nu$ for $\mu \neq \nu$. Let $\Xi_0$ be the quotient space of $\Xi$ with respect to multiplication by a nonzero constant. A point of $\Xi_0$ can be represented by a sequence $\{c, 1, -c, -1\}$ such that $c \neq 0,\ \infty, \pm 1$. Let $b = c^2 \neq 0,\ \infty,\ 1$. The fundamental group of $\overline{\mathbf{C}} \setminus \{0,\ \infty,\ 1\}$ with a base point $-1$ is a free group with two generators $s_0$ and $s_\infty$ corresponding to the loops starting at $-1$, going along the negative real axis towards 0 (resp. $\infty$), going counterclockwise about 0 (resp. $\infty$) and returning to $-1$ along the negative real axis. Each of these generators defines two paths in the space $\Xi_0$ (denoted also by $s_0$ and $s_\infty$) starting at $c = i$ (resp. $c = -i$) and ending at $c = -i$ (resp. $c = i$) (see Fig. 5).

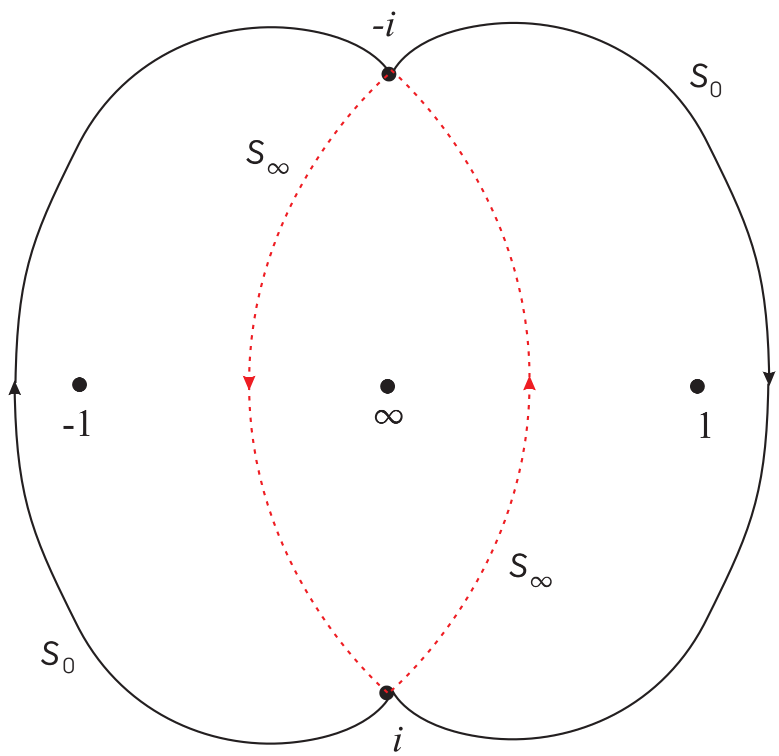


Fig. 5. Paths $s_0$ and $s_\infty$.

The fundamental group of $\Xi_0$ with the base point $i$ is generated by $q_0 = s_0^2$, $q_\infty = s_\infty^2$, $q_{-1} = (s_0 s_\infty)^{-1}$, and $q_1 = (s_\infty s_0)^{-1}$, with the relation $q_0 q_1 q_\infty q_{-1} = 1$.

Suppose that $f$ has nonzero asymptotic values $c_1 = i$, $c_2 = 1$, $c_3 = -i$, $c_4 = -1$. Define four loops $\gamma_i$, $\gamma_1$, $\gamma_{-i}$, $\gamma_{-1}$ in $\mathbf{C}_w^\bullet$ by following either real or imaginary axis from $\infty$ to one of the points $i$, $1$, $-i$, $-1$, moving about that point counterclockwise, and returning to $\infty$ along the same axis (Fig. 1). These four loops generate the free group $\pi_1(\mathbf{C}_w^\bullet \setminus \{\pm i, \pm 1\})$. We can assume that the cell decomposition $\Psi_0$ of $\mathbf{C}_w^\bullet$ defined by these four loops is both centrally symmetric and invariant with respect to complex conjugation. The cyclic order of the loops $\gamma_i$, $\gamma_1$, $\gamma_{-i}$, $\gamma_{-1}$ at the point $\infty$ agrees with the cyclic order of the corresponding Stokes sectors in $\mathbf{C}_z$. According to propositions 6 and 7, the cell decomposition $\Psi_f$ of $\mathbf{C}_z$ can be defined by a tree $T$.

Each path $\{c(t), 1, -c(t), -1\}$ in $\Xi_0$ starting at $c(0) = i$ defines a continuous deformation
$\gamma_i(t)$, $\gamma_1(t)$, $\gamma_{-i}(t)$, $\gamma_{-1}(t)$ of the original four loops, each of the deformed loops starting and ending at $\infty$ and avoiding $0$, $\infty$, $\pm 1$, $\pm c(t)$. This deformation is unique up to isotopy. We can choose the deformation so that the corresponding cell decomposition $\Psi_0(t)$ of $\mathbf{C}_w^\bullet$ remains centrally symmetric. For the paths corresponding to $s_0$ and $s_\infty$, we have $c(1) = -i$. Hence the loops $\hat\gamma_{-i} = \gamma_i(1)$, $\hat\gamma_1 = \gamma_1(1)$, $\hat\gamma_i = \gamma_{-i}(1)$, $\hat\gamma_{-1} = \gamma_{-1}(1)$ belong to the same space $\mathbf{C}_w^\bullet \setminus \{\pm i, \pm 1\}$ as the original loops $\gamma_i$, $\gamma_1$, $\gamma_{-i}$, $\gamma_{-1}$. See Figs. 6, 7.

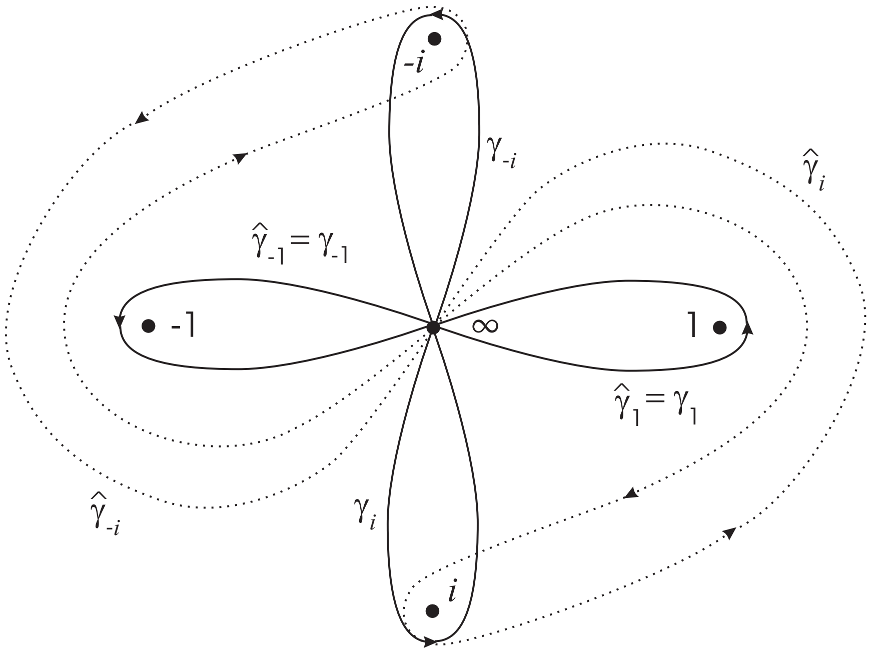


Fig. 6. Action of $s_0$.

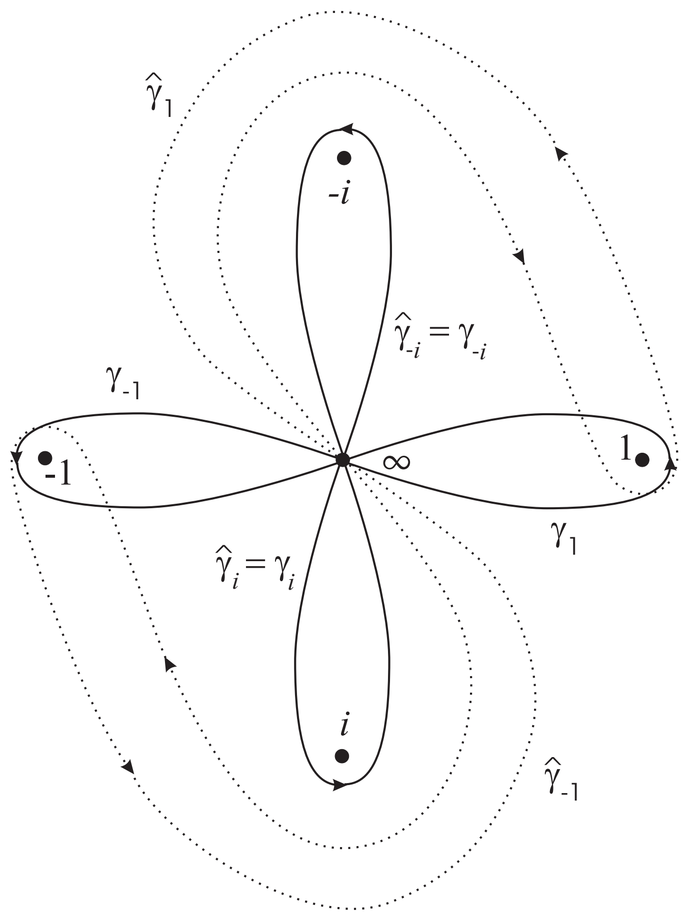


Fig. 7. Action of $s_\infty$.

The paths in $\Xi_0$ corresponding to $s_0$ and $s_\infty$ can be considered as elements of the braid group $\mathcal{B}_4$ on four strands in $\mathbf{C}_w^\bullet$ (leaving $\pm 1$ fixed). From the classical formulas for the action of $\mathcal{B}_k$ on the fundamental group of the plane without $k$ points [23], we have (see Fig. 6 for $s_0$ and Fig. 7 for $s_\infty$).

$$\hat{\gamma}_i = (\gamma_1)^{-1}\gamma_i\gamma_1,\ \hat{\gamma}_1 = \gamma_1,\ \hat{\gamma}_{-i} = (\gamma_{-1})^{-1}\gamma_{-i}\gamma_{-1},\ \hat{\gamma}_{-1} = \gamma_{-1} \text{ for } s_0; \tag{14}$$

$$\hat{\gamma}_i = \gamma_i,\ \hat{\gamma}_1 = (\gamma_{-i})^{-1}\gamma_1\gamma_{-i},\ \hat{\gamma}_{-i} = \gamma_{-i},\ \hat{\gamma}_{-1} = (\gamma_i)^{-1}\gamma_{-1}\gamma_i, \text{ for } s_\infty. \tag{15}$$

Conversely, the original loops can be expressed as products of the new loops:

$$\gamma_i = \hat{\gamma}_1\hat{\gamma}_i(\hat{\gamma}_1)^{-1},\ \gamma_1 = \hat{\gamma}_1,\ \gamma_{-i} = \hat{\gamma}_{-1}\hat{\gamma}_{-i}(\hat{\gamma}_{-1})^{-1},\ \gamma_{-1} = \hat{\gamma}_{-1} \text{ for } s_0; \tag{16}$$

$$\gamma_i = \hat\gamma_i,\ \gamma_1 = \hat\gamma_{-i}\hat\gamma_1(\hat\gamma_{-i})^{-1},\ \gamma_{-i} = \hat\gamma_{-i},\ \gamma_{-1} = \hat\gamma_i\hat\gamma_{-1}(\hat\gamma_i)^{-1},\ \text{for } s_\infty. \tag{17}$$

Let $f_t$ be the family of functions constructed in Proposition $3'$, for the path $\{c(t), 1, -c(t), -1\}$ in $\Xi_0$ with $c(0) = i$ and $c(1) = -i$, and let $\Psi_{f_t} = f_t^{-1}(\Psi_0(t))$ be the corresponding cell decompositions. Note that the cell decomposition $\Psi_0(1)$ is defined by the loops $\hat\gamma_{-i} = \gamma_i(1)$, $\hat\gamma_1 = \gamma_1(1)$, $\hat\gamma_i = \gamma_{-i}(1)$, $\hat\gamma_{-1} = \gamma_{-1}(1)$. Then $f_1$ has the nonzero asymptotic values $-i,\ 1,\ i,\ -1$. The cell decomposition $\Psi_{f_1}$ is obtained by a continuous deformation exchanging $i$ and $-i$ from the cell decomposition $\Psi_{f_0}$. Accordingly, the directed graph without loops $\hat\Gamma$ corresponding to $\Psi_{f_1}$ is the same as the graph $\Gamma$ corresponding to $\Psi_{f_0}$, with the labels $i$ and $-i$ of its edges exchanged.

Let $\Psi'_{f_1} = f_1^{-1}(\Psi_0)$, and let $\Gamma'$ be the corresponding directed graph without loops. Embedding of the graph $\Gamma'$ in the plane $\mathbf{C}_z$ is determined, up to an orientation-preserving homeomorphism of $\mathbf{C}_z$, by its combinatorial structure and the cyclic order of its labeled directed edges at each of its vertices of degree greater than 2 [23]. Since this cyclic order agrees with the cyclic order of directed edges of the cell decomposition $\Psi_0$ at its vertex $\infty$, to define $\Gamma'$ it is enough to specify its combinatorial structure, i.e., for a vertex $v$ of $\Gamma'$ (which can be identified with a vertex of $\hat\Gamma$, since both graphs have $f_1^{-1}(\infty)$ as their vertices) to determine whether an edge of $\Gamma'$ labeled with one of $i,\ 1,\ -i,\ -1$ exits $v$, and if yes, which vertex of $\Gamma'$ does it enter.

Two vertices $v$ and $v'$ of $\hat\Gamma$ are connected by a directed edge labeled by $i$ (resp., $1,\ -i,\ -1$) if and only if the monodromy of $f_1^{-1}$ along the loop $\hat\gamma_i$ (resp., $\hat\gamma_1,\ \hat\gamma_{-i},\ \hat\gamma_{-1}$) maps $v$ to $v' \neq v$. For $\Gamma'$, the same holds for the monodromy of $f_1^{-1}$ along the loop $\gamma_i$ (resp., $\gamma_1,\ \gamma_{-i},\ \gamma_{-1}$).

Let us denote the corresponding monodromy transformations by $\sigma_i,\ \sigma_1,\ \sigma_{-i},\ \sigma_{-1}$, and by $\hat\sigma_i,\ \hat\sigma_1,\ \hat\sigma_{-i},\ \hat\sigma_{-1}$, respectively. Then the transformations $\hat\sigma$ can be determined from the transformations $\sigma$ from the relations (16) and (17) between the two sets of loops. Note that the monodromy is anti-representation of the fundamental group, i.e., the transformation corresponding to the product $\gamma\gamma'$ of two elements of the fundamental group is $\sigma' \circ \sigma$, where $\sigma$ and $\sigma'$ are the monodromy transformations corresponding to $\gamma$ and $\gamma'$.

For $s_0$, the edges of $\Gamma'$ labeled with 1 and $-1$ are the same as the edges of $\hat\Gamma$ with the same labels. For each vertex $v$, the edge of $\Gamma'$ labeled with $i$ that starts at $v$ ends at the vertex obtained from $v$ by moving along an edge of $\hat\Gamma$ labeled with 1 (or staying at $v$ if there is no such edge), then along an edge of $\hat\Gamma$ labeled with $i$, then backward along an edge of $\hat\Gamma$ labeled with 1.

(There is no edge when the last vertex coincides with $v$.)

The edge of $\Gamma'$ labeled with $-i$ that starts at $v$ ends at the vertex obtained from $v$ by moving along an edge of $\hat{\Gamma}$ labeled with $-1$, then along an edge of $\hat{\Gamma}$ labeled with $-i$, then backward along an edge of $\hat{\Gamma}$ labeled with $-1$.

For $s_\infty$, the edges of $\Gamma'$ labeled with $i$ and $-i$ are the same as the edges of $\hat{\Gamma}$ with the same labels. For each vertex $v$, the edge of $\Gamma'$ labeled with 1 that starts at $v$ ends at the vertex obtained from $v$ by moving along an edge of $\hat{\Gamma}$ labeled with $-i$, then along an edge of $\hat{\Gamma}$ labeled with 1, then backward along an edge of $\hat{\Gamma}$ labeled with $-i$. The edge of $\Gamma'$ labeled with $-1$ that starts at $v$ ends at the vertex obtained from $v$ by moving along an edge of $\hat{\Gamma}$ labeled with $i$, then along an edge of $\hat{\Gamma}$ labeled with $-1$, then backward along an edge of $\hat{\Gamma}$ labeled with $i$.

Fig. 8 shows the action of $s_0$ for the graph $\Gamma$ corresponding to the tree $T$ of type $A_1$. The graph $\Gamma'$ corresponds to an undirected graph $T'$ of type $Q_{1,0}$.

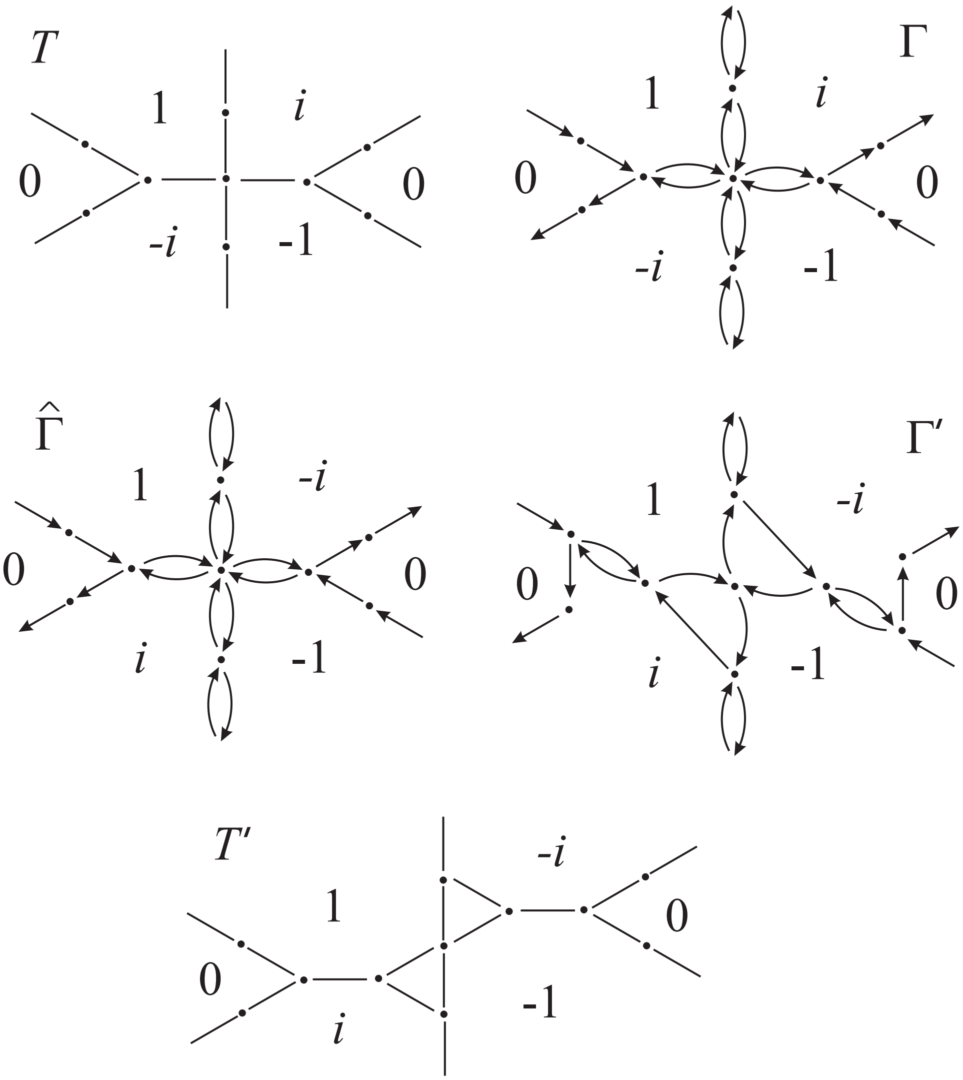


Fig. 8. Action of $s_0$ transforms $A_1$ to $Q_{1,0}$.

Similarly, for the paths $(s_0)^{-1}$ and $(s_\infty)^{-1}$, the transformations $\hat{\sigma}$ can be determined from the transformations $\sigma$ from the relations (14) and (15). However, we do not need this, since they can also be obtained from the symmetry with respect to complex conjugation, sending a function $f(z)$ to $\bar{f}(\bar{z})$. Since the cell decomposition $\Psi_0$ in Fig. 1 is symmetric with respect to complex conjugation, the cell decomposition $\Psi_f$ and the corresponding graph $\Gamma$ is exchanged its complex conjugate, $c$ with $1/\bar{c}$, $\alpha$ with $\bar{\alpha}$, and the action of $s_0$ with the action of $s_\infty^{-1}$.

The action of $s_0$ and $s_\infty$ and their inverses on all trees of Proposition 8 is summarized in the following tables (see also Figs. 9, 10, where this action is represented graphically).

| | |
|---|---|
| $s_0(A_k) = Q_{k,0},\ k \geq 0;$ | $s_0(Q_{k,0}) = A_{k+1},\ k \geq 0;$ |
| $s_0(D_{k,l}) = Q_{k,l},\ k,l \geq 0;$ | $s_0(Q_{k,l}) = D_{k+1,l},\ k,l \geq 0;$ |
| $s_0(E_{k,l}) = E'_{k,l},\ k \geq 1,\ l \geq 0;$ | $s_0(E'_{k,l}) = E_{k-1,l},\ k \geq 1,\ l \geq 0;$ |
| $s_0(\bar{D}_{k,l}) = \bar{Q}_{k,l-2},\ k \geq 0,\ l \geq 2;$ | $s_0(\bar{Q}_{k,l}) = \bar{D}_{k,l+2},\ k,l \geq 0;$ |
| $s_0(\bar{E}_{k,l}) = \bar{E}'_{k,l},\ k \geq 1,\ l \geq 0;$ | $s_0(\bar{E}'_{k,l}) = \bar{E}_{k,l},\ k \geq 1,\ l \geq 0;$ |
| $s_0(\bar{D}_{k,1}) = O_k,\ k \geq 0$ | $s_0(O_k) = \bar{D}_{k,1},\ k \geq 0;$ |
| $s_0(E'_{1,l}) = D_{0,l},\ l \geq 0;$ | $s_0(E'_{1,0}) = A_0.$ |

Table 1. The action of $s_0$.

| | |
|---|---|
| $s_\infty(A_k) = \bar{Q}_{k-1,0},\ k \geq 1;$ | $s_\infty(\bar{Q}_{k,0}) = A_k,\ k \geq 0;$ |
| $s_\infty(D_{k,l}) = Q_{k,l-2},\ k \geq 0,\ l \geq 2;$ | $s_\infty(Q_{k,l}) = D_{k,l+2},\ k,l \geq 0;$ |
| $s_\infty(E_{k,l}) = E'_{k,l},\ k \geq 1,\ l \geq 0;$ | $s_\infty(E'_{k,l}) = E_{k,l},\ k \geq 1,\ l \geq 0;$ |
| $s_\infty(\bar{D}_{k,l}) = \bar{Q}_{k-1,l},\ k \geq 1,\ l \geq 1;$ | $s_\infty(\bar{Q}_{k,l}) = \bar{D}_{k,l},\ k \geq 1,\ l \geq 1;$ |
| $s_\infty(\bar{E}_{k,l}) = \bar{E}'_{k+1,l},\ k \geq 1,\ l \geq 0;$ | $s_\infty(\bar{E}'_{k,l}) = \bar{E}_{k,l},\ k \geq 1,\ l \geq 0;$ |
| $s_\infty(D_{k,1}) = O_k,\ k \geq 0;$ | $s_\infty(O_k) = D_{k,1},\ k \geq 0;$ |
| $s_\infty(\bar{D}_{0,l}) = \bar{E}'_{1,l},\ l \geq 0;$ | $s_\infty(A_0) = \bar{E}'_{1,0}.$ |

Table 2. The action of $s_\infty$.

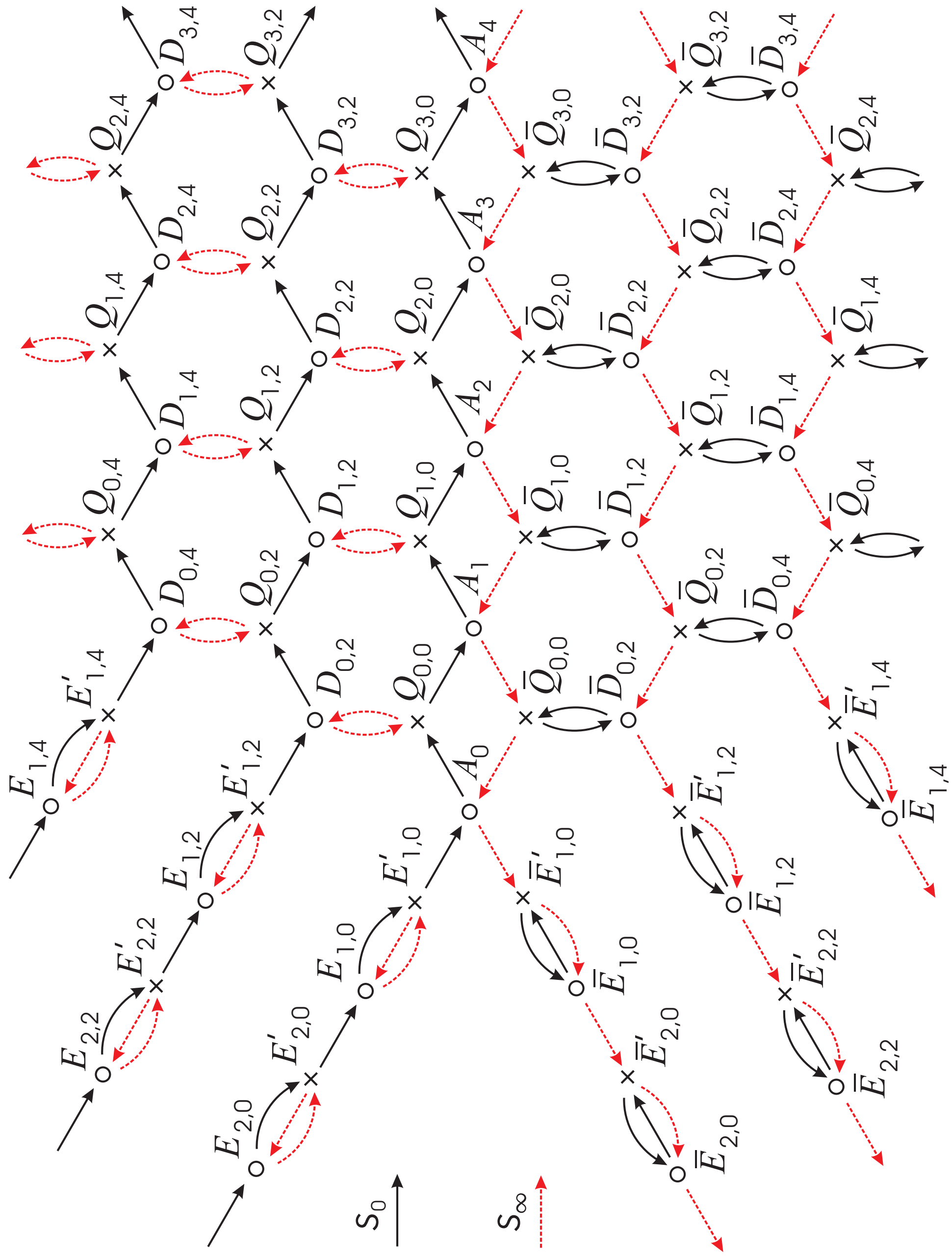


Fig. 9. Monodromy action on the even part of $G$.

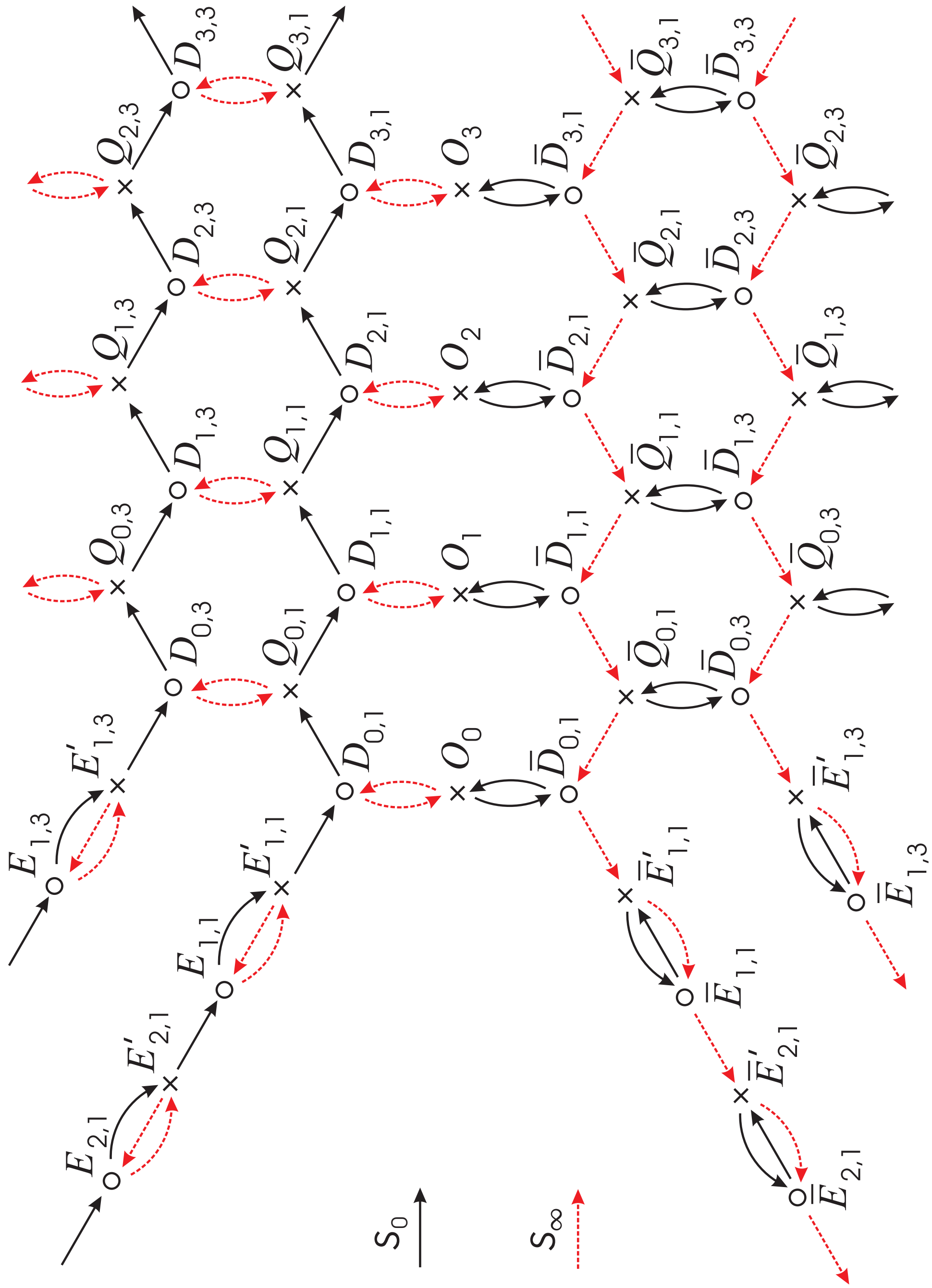


Fig. 10. Monodromy action on the odd part of $G$.

*Completion of the proof of Theorem 1 a).*

To prove that the Riemann surface $G$ has exactly two connected components, corresponding to functions with a pole at the origin and the functions with a zero at the origin, respectively, it is enough to show that, for each point $c_0 \in \overline{\mathbf{C}}$ over which the mapping $W : G \to \overline{\mathbf{C}}$ is not ramified (i.e., $c_0 \neq 0, \infty, \pm 1$) the monodromy of $W^{-1}$ acts transitively on the fiber $W^{-1}(c_0)$ of $W$. Let us choose a point $c_0 = i$. With the loops $\gamma_i, \gamma_1, \gamma_{-i}, \gamma_{-1}$ defined above, the fiber of $W^{-1}(i)$ consists of the functions $f$ such that the cell decomposition $\Psi_f$ of $\mathbf{C}_z$ corresponds to one of the trees $A_k,\ D_{k,l},\ E_{k,l}, \bar{D}_{k,l},\ \bar{E}_{k,l}$. From the tables for the action of $s_0$ and $s_\infty$, for the trees with a vertex at the origin we have:
**(i)** $A_k$ can be obtained from $A_0$ applying $(s_0)^{2k}$;
**(ii)** $D_{k,2l}$ can be obtained from $A_k$ applying $(s_0 s_\infty)^l$;
**(iii)** $E_{k,2l}$ can be obtained from $D_{0,2l}$ applying $(s_0)^{-2k}$;
**(iv)** $E_{k,0}$ can be obtained from $A_0$ applying $(s_0)^{-2k}$;
**(v)** The points corresponding to $\bar{D}_{k,2l}$ and $\bar{E}_{k,2l}$ can be obtained from $A_0$ combining the paths in (i-iv) with the complex conjugation.

For the trees with no vertex at the origin we have:
**(i)** $D_{k,1}$ can be obtained from $D_{0,1}$ applying $(s_0)^{2k}$;
**(ii)** $D_{k,2l+1}$ can be obtained from $D_{k,1}$ applying $(s_0 s_\infty)^l$;
**(iii)** $E_{k,2l+1}$ can be obtained from $D_{0,2l+1}$ applying $(s_0)^{-2k}$;
**(iv)** $\bar{D}_{k,1}$ can be obtained from $D_{k,1}$ applying $s_\infty s_0$;
**(v)** The points corresponding to $\bar{D}_{k,2l+1}$ and $\bar{E}_{k,2l+1}$ can be obtained from $D_{0,1}$ combining the paths in (i-iv) with the complex conjugation.

*Eigenfunctions of self-adjoint operators.*

Suppose that $\alpha$ in (4) is real, i.e., the problem is self-adjoint and the eigenvalues are real. Let $\lambda_0 < \lambda_1 < \ldots$ be the eigenvalues of (4) and $y_1(z),\ y_2(z), \ldots$ the corresponding eigenfunctions. Then $y_n(z)$ has $n$ real zeros (see [31]). Let $\lambda = \lambda_n$ be one of these eigenvalues. Since $\alpha$ and $\lambda$ are real, the function $f(z)$ defined in Section 2 is a real odd meromorphic function. Hence its nonzero asymptotic values satisfy $c_2 = -\bar{c}_1,\ c_3 = -c_1,\ c_4 = \bar{c}_1$. These asymptotic values can be neither real nor pure imaginary (see [16]). Let $a = c_1$, and let $\Psi_a$ be a cell decomposition of $\overline{\mathbf{C}}_w$ defined similarly to the cell decomposition $\Psi_0$ in Fig. 1, except the four loops of $\Psi_a$ contain the asymptotic values $\pm a,\ \pm \bar{a}$ of $f$ (see Fig. 11).

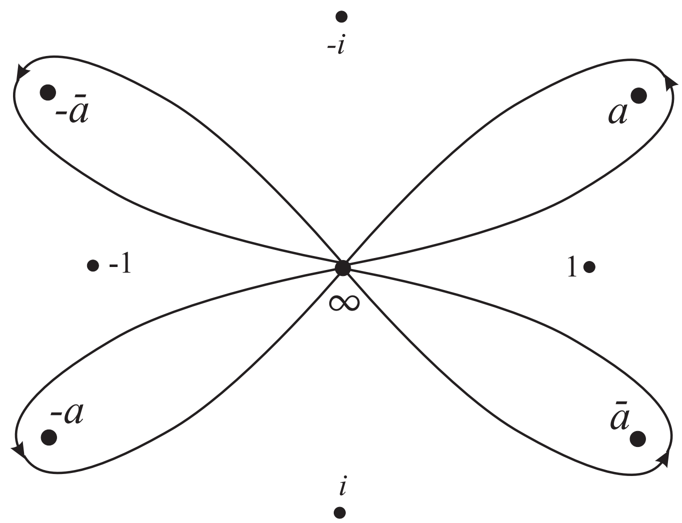


Fig. 11. Cell decomposition for a real function.

We assume that $\Psi_a$ is centrally symmetric and invariant under complex conjugation. Then the cell decomposition $\Psi_f = f^{-1}(\Psi_a)$ of $\mathbf{C}_z$ is also centrally symmetric and invariant under complex conjugation.

**Proposition 9.** *The cell decomposition $\Psi_f$ is of the type $A_k$ for $n = 2k$, and of the type $O_k$ for $n = 2k + 1$.*

*Proof.* Since $\Psi_f$ is invariant under complex conjugation in $\mathbf{C}_z$, the corresponding graphs $\Gamma$ and $T$ are symmetric under complex conjugation.

If $a$ is in the second or fourth quadrant (as in Fig. 11) the cyclic order of the nonzero asymptotic values of $f$ in $\mathbf{C}_w^\bullet$ is consistent with the cyclic order of the corresponding Stokes sectors $S_1$, $S_2$, $S_4$, $S_5$. Thus conditions of Proposition 6 are satisfied, and the graph $T$ should be one of the trees listed in Proposition 8. Since the only such trees symmetric under complex conjugation are $A_k$, we have $T = A_k$ for some $k$. A real function $f$ with the cell decomposition $\Psi_f$ of the type $A_k$ has $2k$ real zeros (one zero in each two-gon of $\Psi_f$ corresponding to an edge of $T$ on the real line). Hence $n = 2k$ in this case.

If $a$ is either in the first or third quadrant, the cyclic order of the asymptotic values of $f$ is opposite to the cyclic order of the corresponding Stokes sectors. In this case, operations $s'_0$ and $s'_\infty$ similar to $s_0$ and $s_\infty$ (Figs. 6 and 7) can be applied to reverse the cyclic order. More precisely, for $a = \pm e^{i\pi/4}$, the function $g(z) = e^{i\pi/4} f(z)$ has the asymptotic values $\pm 1$, $\pm i$ and the cell decomposition $\Psi_g = g^{-1}(\Psi_0)$, where $\Psi_0$ is the cell decomposition in Fig. 1, coincides with $\Psi_f$. The action of $s_0'$ and $s_\infty'$ on $\Psi_f$ is defined as the action of $s_0$ and $s_\infty$ on $\Psi_g$. For any $a$ in the first or third quadrant, one can deform $\pm a$ to $\pm e^{i\pi/4}$ without crossing the real and imaginary exes, apply the two operations, and deform $\pm a$ back to initial values. Thus the action of $s_0'$ and $s_\infty'$ on $\Psi_f$ is the same as the action of $s_0$ and $s_\infty$ described in Tables 1 and 2. In particular, the result of this action is a cell decomposition satisfying the condition of Proposition 6, hence having the type of one of the trees listed in Proposition 8. Accordingly, the cell decomposition $\Psi_f$ itself should correspond to a graph $T$ obtained from one of these trees by the operations $s_0^{-1}$ and $s_\infty^{-1}$. From Tables 1 and 2, all such graphs are of the types $O_k,\ Q_{k,l},\ E'_{k,l},\ \bar{Q}_{k,l},\ \bar{E}'_{k,l}$. The only graphs in this list symmetric under complex conjugation are $O_k$. A real function $f$ with the cell decomposition $\Psi_f$ of the type $O_k$ has $2k+1$ real zeros (one zero at the origin and one inside each two-gon of $\Psi_f$ corresponding to an edge of $T$ on the real line). Hence $n = 2k+1$ in this case.

## 6. Other potentials

The following three one-parametric families of potentials can be treated with the same method. The details will appear elsewhere.

1. PT-symmetric cubic [11]. A differential equation $-y'' + P(z)y = \lambda y$, with a general cubic polynomial $P$, by an affine change of the independent variable and a shift of $\lambda$ can be brought to the form

$$-y'' + (iz^3 + i\alpha z)y = \lambda y. \tag{18}$$

We impose the boundary condition

$$y(z) \to 0, \quad \text{as} \quad y \in \mathbf{R}, \quad y \to \pm\infty. \tag{19}$$

This problem is not self-adjoint but for real $\alpha$ it has the so-called PT-symmetry property. It is known [12, 30] that for $\alpha \geq 0$ the spectrum is real.

**Theorem 3.** *Let $Z_3 \in \mathbf{C}^2$ be the set of all pairs $(\alpha, \lambda)$ such that the problem (18), (19) has a non-trivial solution. Then $Z_3$ is an irreducible non-singular curve.*

2. Quasi-exactly solvable sextic family [36, 37]. Consider the equation

$$-y'' + (z^6 + 2\alpha z^4 + \{\alpha^2 - (4m + 2p + 3)\}z^2)y = \lambda y, \tag{20}$$

with the same boundary condition (19). This problem is self-adjoint for real $\alpha$. It was shown by Turbiner and Ushveridze that for real $\alpha$ this problem has exactly $m+1$ linearly independent "elementary" eigenfunctions of the form $Qe^T$ with polynomials $Q$ and $T$. The degree of $Q$ is $2m+p$, so the eigenfunction has $2m+p$ zeros in the complex plane. If $p=0$ then these elementary eigenfunctions correspond to the first $m+1$ even-numbered eigenvalues, and if $p=1$ to the first $m+1$ odd-numbered eigenvalues.

**Theorem 4.** *Let $m$ be a non-negative integer and $p \in \{0, 1\}$. Let $Z_{6,m,p} \in \mathbf{C}^2$ be the set of all pairs $(\alpha, \lambda)$ such that $\lambda$ is an eigenvalue of the problem (20), (19) corresponding to an elementary eigenfunction. Then $Z_{6,m,p}$ is a non-singular irreducible curve.*

3. Quasi-exactly solvable PT-symmetric quartic family [3]. Consider the equation

$$-y'' + (-z^4 - 2\alpha z^2 - 2imz)y = \lambda y. \tag{21}$$

Here the boundary condition is

$$y(re^{i\theta}) \to 0, \quad \text{as} \quad r \to \infty, \quad \theta \in \{-\pi/6, -\pi + \pi/6\}. \tag{22}$$

Similarly to the previous example, this problem has $m$ elementary eigenfunctions.

**Theorem 5.** *Let $m$ be a non-negative integer, and $Z_{4,m} \subset \mathbf{C}^2$ be the set of pairs $(\alpha, \lambda)$ such that $\lambda$ is an eigenvalue of the problem (21), (22), with an elementary eigenfunction. Then $Z_{4,m}$ is a smooth irreducible curve.*

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