

# Green's function and anti-holomorphic dynamics on a torus

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## Abstract

We give a new, simple proof of the fact recently discovered by C.-S. Lin and C.-L. Wang that the Green function of a torus has either three or five critical points, depending on the modulus of the torus. The proof uses anti-holomorphic dynamics. As a byproduct we find a one-parametric family of anti-holomorphic dynamical systems for which the parameter space consists only of hyperbolic components and analytic curves separating them.

## 1 Introduction

Green's function on a torus  $T$  is defined as a solution of the equation

$$\Delta G = -\delta + \frac{1}{|T|},$$

normalized so that

$$\int_T G = 0.$$

Here  $\delta$  is the delta-function, and  $|T|$  is the area of  $T$  with respect to a flat metric.

We write the torus  $T$  as  $T = \mathbb{C}/\Lambda$  with a lattice

$$\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\},$$

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where  $\tau = \omega_2/\omega_1$  satisfies  $\text{Im } \tau > 0$ . Recently C.-S. Lin and C.-L. Wang [23] discovered that Green's function has either three or five critical points, depending on  $\tau$ . It is surprising that this simple fact was not known until 2010. In [8, 9] they study the corresponding partition of the  $\tau$ -half-plane. Their proofs are long and indirect, using advanced non-linear PDE theory and modular forms. Our paper is motivated by the desire to give a simple proof of their result that Green's function has either three or five critical points and to give a criterion for  $\tau$  distinguishing which case occurs.

We have (see [23])

$$G(z) = -\frac{1}{2\pi} \log |\theta_1(z)| + \frac{(\text{Im } z)^2}{2 \text{Im } \tau} + C(\tau),$$

where  $\theta_1$  is the first theta-function. Here and in the following we use the notation of elliptic functions as given in [1, 16]. We note that the notation in [2, 28, 29] is different, see the remark following the theorem below.

Critical points of  $G$  are solutions of the equation

$$\zeta(z) + az + b\bar{z} = 0, \tag{1}$$

where the constants  $a$  and  $b$  are uniquely defined by the condition that the left hand side is  $\Lambda$ -periodic. With  $\zeta(z + \omega_j) = \zeta(z) + \eta_j$  for  $j = 1, 2$  we thus have

$$\eta_1 + a\omega_1 + b\bar{\omega}_1 = 0 \quad \text{and} \quad \eta_2 + a\omega_2 + b\bar{\omega}_2 = 0.$$

With the Legendre relation  $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$  we obtain

$$b = -\frac{\pi}{|\omega_1|^2 \text{Im } \tau} \quad \text{and} \quad a = -\frac{b\bar{\omega}_1}{\omega_1} - \frac{\eta_1}{\omega_1} = \frac{\pi}{\omega_1^2 \text{Im } \tau} - \frac{\eta_1}{\omega_1}. \tag{2}$$

So the problem is to determine the number of solutions of (1) where  $a$  and  $b$  are given by (2).

**Theorem.** *The equation (1) has three solutions in  $T$  if  $e_j\omega_1^2 + \eta_1\omega_1 = 0$  or*

$$\text{Im} \left( \frac{2\pi i}{e_j\omega_1^2 + \eta_1\omega_1} - \tau \right) \geq 0 \tag{3}$$

*for some  $j \in \{1, 2, 3\}$  and it has five solutions otherwise.*

Here, as usual,  $e_1 = \wp(\omega_1/2)$ ,  $e_2 = \wp((\omega_1 + \omega_2)/2)$  and  $e_3 = \wp(\omega_2/2)$ . An elementary computation shows that the condition in the theorem is equivalent to

$$\min_{1 \leq j \leq 3} \left| \frac{e_j \omega_1^2 + \eta_1 \omega_1}{\pi} \operatorname{Im} \tau - 1 \right| \leq 1$$

We note that  $e_j \omega_1^2$  and  $\eta_1 \omega_1$  depend only on  $\tau = \omega_2/\omega_1$ . We may restrict to the case that  $\omega_1 = 1$  so that  $\tau = \omega_2$ . Then (3) simplifies to

$$\operatorname{Im} \left( \frac{2\pi i}{e_j + \eta_1} - \tau \right) \geq 0.$$

As mentioned, a different notation for elliptic functions is used in [2, 28, 29]. There the periods are denoted by  $2\omega_j$  and the definition of  $\eta_1$  also differs by a factor 2. Thus in that terminology (3) takes the form

$$\operatorname{Im} \left( \frac{\pi i}{2(e_j \omega_1^2 + \eta_1 \omega_1)} - \tau \right) \geq 0.$$

Figure 1 shows (in gray) the regions in the  $\tau$ -plane where Green's function has 5 critical points; that is, the set of  $\tau$ -values where (3) fails for all  $j$ . The range shown is  $|\operatorname{Re} \tau| \leq 1$  and  $0.15 \leq \operatorname{Im} \tau \leq 2.15$ .

The standard fundamental domain consisting of those  $\tau$  which satisfy the inequalities  $\operatorname{Im} \tau > 0$ ,  $-1/2 < \operatorname{Re} \tau \leq 1/2$  and  $|\tau| \geq 1$ , with  $|\tau| = 1$  only if  $\operatorname{Re} \tau \geq 0$ , is in the upper middle of the picture. Its images under the modular group are also shown.

Our proof is based on Fatou's theorem from complex dynamics. Originally Fatou's theorem was proved to estimate from above the number of attracting cycles of a rational function. Then it was extended to more general classes of functions. The most surprising fact is that Fatou's theorem can be used sometimes to estimate the number of solutions of equations in settings where dynamics is not present. This was first noticed in [11]; the contents of this unpublished preprint is reproduced in [15, 7]. The paper [5] shows that Fatou's theorem can be used to prove under some circumstances the existence of critical points of a meromorphic function. In the papers [20, 18, 19, 17] Fatou's theorem was used to obtain upper estimates of the numbers of solutions of equations of the form

$$z = r(\bar{z}),$$

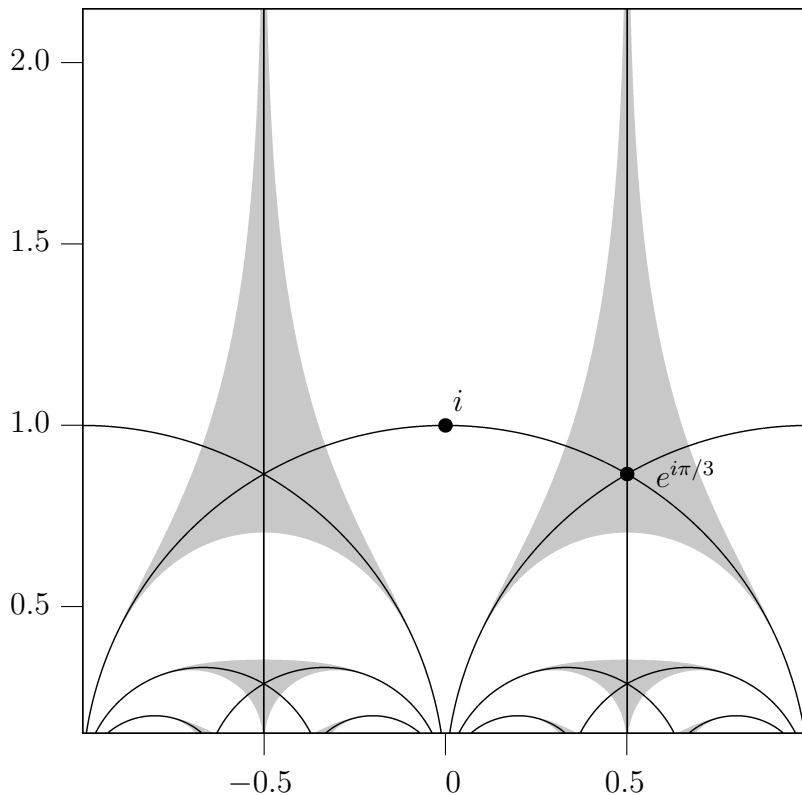


Figure 1: The regions given by (3) in the  $\tau$ -plane.

with a meromorphic function  $r$ ; this permitted to prove a conjecture in astronomy [18, 19]. In the recent work [21], a topological classification of quadrature domains is obtained with a method based on Fatou's theorem.

As mentioned, Fatou [12, §30] stated his result originally only for rational functions. As pointed out for example in [4, Theorem 7], the proof extends to functions meromorphic in  $\mathbb{C}$ . Here we need a slightly more general version, namely for functions meromorphic in  $\overline{\mathbb{C}} \setminus E$ , where  $E$  is a countable compact subset of  $\overline{\mathbb{C}}$ . For example, iterates of functions meromorphic in  $\mathbb{C}$  are of this form. Again Fatou's proof extends to this context; see [3, Lemma 10], where in fact a more general class of functions is considered.

To state the version of Fatou's theorem that we need, let  $E$  be a countable compact subset of  $\overline{\mathbb{C}}$  and let  $f: \overline{\mathbb{C}} \setminus E \rightarrow \overline{\mathbb{C}}$  be meromorphic, and suppose that  $f$  does not extend meromorphically to any point of  $E$ . For simplicity we also

assume that  $\infty \in E$ , as this will be the case in our situation.

A point  $z_0 \in \mathbb{C} \setminus E$  is called *fixed* if  $f(z_0) = z_0$ , and for such a point  $f'(z_0)$  is called the *multiplier* of  $z_0$ . A fixed point is called *attracting*, *neutral* or *repelling* depending on whether the modulus of its multiplier is less than, equal to or greater than 1, respectively. A neutral fixed point is called *rational* if the multiplier is a root of unity.

A point  $c \in \mathbb{C} \setminus E$  is called a *critical* point of  $f$  if  $f'(c) = 0$  and it is called an *asymptotic value* of  $f$  if there exists a curve  $\gamma: [0, 1) \rightarrow \mathbb{C} \setminus E$  such that  $\gamma(t)$  tends to a point in  $E$  as  $t \rightarrow 1$  while  $f(\gamma(t)) \rightarrow c$  as  $t \rightarrow 1$ .

**Fatou's Theorem.** *For each attracting fixed point  $z_0$  of  $f$  there is a critical point or an asymptotic value  $c$  such that  $f^n(c) \rightarrow z_0$  as  $n \rightarrow \infty$ .*

Actually, Fatou's theorem also holds for rational neutral fixed points, and it also extends to periodic points, that is, fixed points of the iterates of  $f$ , but we do not need these results.

## 2 Proof of the Theorem

To prove the theorem we rewrite (1) as a fixed point equation

$$z = -\frac{1}{b} \left( \overline{\zeta(z)} + \overline{az} \right) =: g(z), \quad (4)$$

so  $g$  is an *anti*-meromorphic function in the plane. As the left hand side of (1) is  $\Lambda$ -periodic, we conclude from (4) that

$$g(z + \omega) = g(z) + \omega, \quad \omega \in \Lambda. \quad (5)$$

The function  $g$  does not map the plane into itself because it has poles, so the equation (5) does not permit to define a map of the torus  $T$  into itself. To remedy this, we consider the set  $P_0$  of poles of  $g$ . For  $n \geq 1$  we define inductively  $P_n = g^{-1}(P_{n-1})$ . Then all iterates of  $g$  are defined outside the closure  $J(g)$  of the union  $P_\infty = \bigcup_{n=0}^\infty P_n$ . In fact, the iterates of  $g$  form a normal family in  $\mathbb{C} \setminus J(g)$ . Thus – with an obvious extension of these concepts from holomorphic functions to anti-holomorphic ones – we call  $J(g)$  the *Julia set* and its complement  $F(g) = \mathbb{C} \setminus J(g)$  the *Fatou set* of  $g$ . The Fatou set is thus the maximal open subset of the plane such that  $g(F(g)) \subset F(g)$ . Evidently,  $F(g)$  is  $\Lambda$ -invariant, so the map  $g: F(g) \rightarrow F(g)$  descends to a

map which is defined on an open subset of the torus  $T$  and maps this open subset to itself.

We apply the terminology attracting, neutral and repelling also to a fixed point  $z_0$  of  $g$ , considering  $\bar{\partial}g(z_0)$  as the multiplier. Note that

$$\overline{\bar{\partial}g(z)} = -\frac{1}{b}(\zeta'(z) + a) = \frac{1}{b}(\wp(z) - a). \quad (6)$$

To obtain holomorphic dynamics instead of the anti-holomorphic one, we consider the second iterate  $h = g^2$ . By the chain rule, we have

$$h' = ((\bar{\partial}g) \circ g) \cdot \bar{\partial}g = ((\bar{\partial}g) \circ g) \cdot \overline{\bar{\partial}g}. \quad (7)$$

For a fixed point  $z_0$  of  $g$  we thus obtain  $h'(z_0) = |\bar{\partial}g(z_0)|^2$ . Even though we will not need this fact, we observe that the multiplier with respect to  $h$  at a fixed point of  $g$  is always a non-negative real number.

In order to apply Fatou's theorem, we have to consider the critical points and asymptotic values of  $h$ .

**Lemma 1.** *The map  $h: \mathbb{C} \setminus P_\infty \rightarrow \mathbb{C}$  has no asymptotic values.*

*Proof.* From (5) we see that  $g(z) \rightarrow \infty$  as  $z \rightarrow \infty$  avoiding small neighborhoods of the poles. So  $g$  cannot have an asymptotic value in  $\mathbb{C}$ , and its second iterate  $h$  also cannot.  $\square$

We consider the equivalence relation defined by  $z \sim z'$  if  $h^n(z) = h^m(z')$  for some non-negative integers  $m$  and  $n$ . The equivalence classes are called the *grand orbits*. We note that if the sequence  $(h^n(z))$  converges for some  $z \in T$ , then for all  $z'$  in the grand orbit of  $z$  the sequence  $(h^n(z'))$  converges to the same limit. We call this the limit of the grand orbit.

**Lemma 2.** *The set of critical points of  $h$  belongs to at most 4 grand orbits under  $h$ .*

*Proof.* By (6), the equation  $\bar{\partial}g(z) = 0$  is equivalent to  $\wp(z) = a$ , and has two solutions which we denote  $c$  and  $-c$ . By (7) the other zeros of  $h'$  are  $g^{-1}(\pm c) = \pm g^{-1}(c)$ . Even though this is an infinite set on  $T$ , the critical points of  $h'$  are thus contained in at most 4 grand orbits represented by  $c$ ,  $-c$ ,  $g(c)$  and  $-g(c)$ .  $\square$

**Lemma 3.** *Let  $z_0$  be a fixed point of  $g$ . If  $h^n(z) \rightarrow z_0$  as  $n \rightarrow \infty$  for some  $z$ , then  $h^n(g(z)) \rightarrow z_0$ .*

*Proof.* We have  $h^n \circ g = g \circ h^n$  for all  $n \in \mathbb{N}$ . Since  $h^n(z) \rightarrow z_0$  this yields  $h^n(g(z_0)) = g(h^n(z)) \rightarrow g(z_0) = z_0$ .  $\square$

Using Fatou's theorem, we deduce the following result from the previous three lemmas.

**Lemma 4.** *The function  $g$  has at most two attracting fixed points.*

The map  $\phi: T \rightarrow \overline{\mathbb{C}}, z \mapsto z - g(z)$ , is well defined by (5). Let  $J_\phi = 1 - |\bar{\partial}g|^2$  be the Jacobian determinant of  $\phi$ . Then  $D^+ = \{z \in T: J_\phi(z) > 0\}$  is the set where  $\phi$  preserves the orientation and  $D^- = \{z \in T: J_\phi(z) < 0\}$  is the set where  $\phi$  reverses the orientation. As  $\phi$  has one pole, a point  $w$  of large modulus has one preimage on  $T$  and the map is reversing orientation at this preimage. We conclude that the degree of  $\phi$  equals  $-1$ ; see [25, §5] for the definition of the degree.

Suppose first that all zeros of  $\phi$  are in  $D^+ \cup D^-$ . Equivalently,  $g$  has no neutral fixed points. Denote by  $N^+$  and  $N^-$  the numbers of zeros of  $\phi$  in  $D^+$  and  $D^-$  respectively. Then  $N^+ - N^-$  equals the degree of  $\phi$  so that  $N^+ - N^- = -1$  and thus

$$N^- = N^+ + 1. \quad (8)$$

For the number  $N = N^+ + N^-$  of fixed points of  $g$  in  $T$  we thus find that

$$N = 2N^+ + 1. \quad (9)$$

Since  $J_\phi = 1 - |\bar{\partial}g|^2$ , the zeros of  $\phi$  in  $D^+$  are attracting fixed points of  $g$  while the zeros of  $\phi$  in  $D^-$  are repelling fixed points of  $g$ . Thus Lemma 4 yields that

$$N^+ \leq 2. \quad (10)$$

It follows from (9) and (10) that  $N \leq 5$ .

On the other hand, since  $\phi$  is odd and  $\Lambda$ -periodic it easily follows that that the half-periods  $\omega_1/2$ ,  $\omega_2/2$  and  $\omega_3/2 = (\omega_1 + \omega_2)/2$  are zeros of  $\phi$ . Equivalently, they are fixed points of  $g$ . Thus we have  $N \geq 3$ . Altogether, since  $N$  is odd by (9), it follows that  $N = 3$  or  $N = 5$ .

It remains to determine the criterion distinguishing the cases. Suppose first that all three half-periods are in  $D^-$ ; that is, they are repelling fixed points of  $g$ . Then  $N^- \geq 3$ . This yields that  $N^- = 3$  and  $N^+ = 2$  so that  $N = 5$  by (8) and (10). Suppose now that one half-period, say  $\omega_j/2$ , is not in  $D^-$  and thus in  $D^+$ . Thus  $\omega_j/2$  is an attracting fixed point of  $g$  and hence

attracts a critical orbit by Fatou's theorem. However, since  $g$  is odd we see that  $\omega_j/2$  in fact attracts both critical orbits. Thus, by Fatou's theorem, there are no other attracting fixed points. Thus  $N^+ = 1$  and hence  $N = 3$  in this case.

We see that  $N = 3$  if and only if there exists  $j \in \{1, 2, 3\}$  such that  $\omega_j/2$  is an attracting fixed point of  $g$ ; that is,  $|\bar{\partial}g(\omega_j/2)| < 1$ . Using (6) this takes the form

$$\min_{1 \leq j \leq 3} \left| \frac{e_j}{b} - \frac{a}{b} \right| < 1. \quad (11)$$

Now (2) yields

$$\frac{a}{b} = -\frac{\bar{\omega}_1}{\omega_1} - \frac{\eta_1}{b\omega_1} = -\frac{\bar{\omega}_1}{\omega_1} + \frac{\eta_1|\omega_1|^2 \operatorname{Im} \tau}{\omega_1 \pi} = \frac{\bar{\omega}_1}{\omega_1} \left( -1 + \frac{\eta_1 \omega_1 \operatorname{Im} \tau}{\pi} \right)$$

and

$$\frac{e_j}{b} = -\frac{e_j|\omega_1|^2 \operatorname{Im} \tau}{\pi} = -\frac{\bar{\omega}_1}{\omega_1} \frac{e_j \omega_1^2 \operatorname{Im} \tau}{\pi}.$$

Substituting the last two equations in (11) yields

$$\left| \frac{e_j \omega_1^2 + \eta_1 \omega_1}{\pi} \operatorname{Im} \tau - 1 \right| < 1.$$

This is equivalent to strict inequality in (3). This completes the proof of the theorem in the case that all zeros of  $\phi$  are in  $D^+ \cup D^-$ .

To deal with the case where this condition is not satisfied, we note that in the above arguments we may replace  $\phi(z)$  by  $\psi(z) = \phi(z) - w$  for any  $w \in \mathbb{C}$ . Noting that  $\phi$  and  $\psi$  have the same Jacobian determinant we conclude that whenever all  $w$ -points of  $\phi$  are in  $D^+ \cup D^-$ , then  $\phi$  has either three or five  $w$ -points in  $T$ . Moreover, there are three  $w$ -points if and only if one them is in  $D^+$ .

We will use the following result [6, Proposition 3].

**Lemma 5.** *Let  $D \subset \mathbb{C}$  be a domain and let  $f: D \rightarrow \mathbb{C}$  be harmonic. Suppose there exists  $m \in \mathbb{N}$  such that every  $w \in \mathbb{C}$  has at most  $m$  preimages. Then the set of points which have  $m$  preimages is open.*

Suppose now that  $\phi$  has a zero in  $T \setminus (D^+ \cup D^-)$ . Then there are arbitrarily small  $w$  such that  $\phi$  has a  $w$ -point in  $D^+$ . Since for such  $w$  the function  $\phi$  has three  $w$ -points, it follows from Lemma 5 that  $\phi$  has three zeros in  $T$ . These zeros are the half-periods and we see that we have equality in (3). This completes the proof of the theorem.

*Remark 1.* The quantities  $e_j\omega_1^2 + \eta_1\omega_1$  occurring in (3) have the following representations via theta functions, see [28, p. 44]:

$$e_1\omega_1^2 + \eta_1\omega_1 = -\frac{\vartheta_2''(0)}{\vartheta_2(0)}, \quad e_2\omega_1^2 + \eta_1\omega_1 = -\frac{\vartheta_0''(0)}{\vartheta_0(0)}, \quad e_3\omega_1^2 + \eta_1\omega_1 = -\frac{\vartheta_3''(0)}{\vartheta_3(0)}.$$

The series for theta functions converge fast and provide a convenient way to compute Figure 1.

Let

$$F_j(\tau) = \frac{2\pi i}{e_j\omega_1^2 + \eta_1\omega_1} - \tau$$

so that (3) takes the form  $\text{Im } F_j(\tau) \geq 0$ . It seems that  $F_j$  maps the components of the set of all  $\tau$  in the upper half-plane where  $\text{Im } F_j(\tau) > 0$  univalently onto the upper half-plane, but we have not been able to prove this.

*Remark 2.* Lin and Wang [23, Theorems 1.6 and 1.7] pay special attention to the case that  $\tau = 1/2 + ib$ , in which case Green's function has five critical points if and only if  $b$  is outside a certain interval  $[b_0, b_1]$  where  $b_0 \approx 0.35$  and  $b_1 \approx 0.71$ .

Here we note that the constants  $b_0$  and  $b_1$  are related to the so-called one-ninth constant  $\Lambda$  occurring in approximation theory; see [13, Section 4.5] for the definition and properties of this constant. As noted there, this constant was already computed by Halphen [14, p. 287] to six digits. It turns out that  $\Lambda = e^{-2\pi b_0} = e^{-\pi/(2b_1)}$ . The numerical values are  $\Lambda = 0.10765391\dots$ ,  $b_0 = 0.35472989\dots$  and  $b_1 = 0.70476158\dots$ .

To prove the above relation between these constants we note first that  $e_1\omega_1^2 + \eta_1\omega_1 = 0$  if and only if  $\vartheta_2''(0) = 0$  and thus if

$$\sum_{k=0}^{\infty} (2k+1)^2 h^{k(k+1)} = 0 \quad \text{where } h = e^{i\pi\tau}.$$

For  $\tau = 1/2 + ib$  with  $b \in \mathbb{R}$  we have  $h = e^{i\pi/2 - \pi b}$  and hence

$$h^{k(k+1)} = \exp\left((i\pi - 2\pi b)\frac{k(k+1)}{2}\right) = (-x)^{k(k+1)/2} \quad \text{with } x = e^{-2\pi b}.$$

For  $\tau = 1/2 + ib$  the condition that  $e_1\omega_1^2 + \eta_1\omega_1 = 0$  is thus equivalent to

$$\sum_{k=0}^{\infty} (2k+1)^2 (-x)^{k(k+1)/2} = 0.$$

The smallest positive solution of the last equation is the one-ninth constant  $\Lambda$ ; see [13, Section 4.5]. Since  $x = e^{-2\pi b}$  we deduce that  $\Lambda = e^{-2\pi b_0}$ .

The modular group leaves the sets where Green's function has five or three critical points invariant. The transformation  $T(z) = (z - 1)/(2z - 1)$  satisfies  $T(1/2 + ib) = 1/2 + i/(4b)$ . This implies that  $b_1 = 1/(4b_0)$  and  $\Lambda = e^{-\pi/(2b_1)}$ .

*Remark 3.* Figure 2 shows the Julia sets of the functions  $g$  corresponding to the square and the hexagonal lattice. The standard fundamental domains

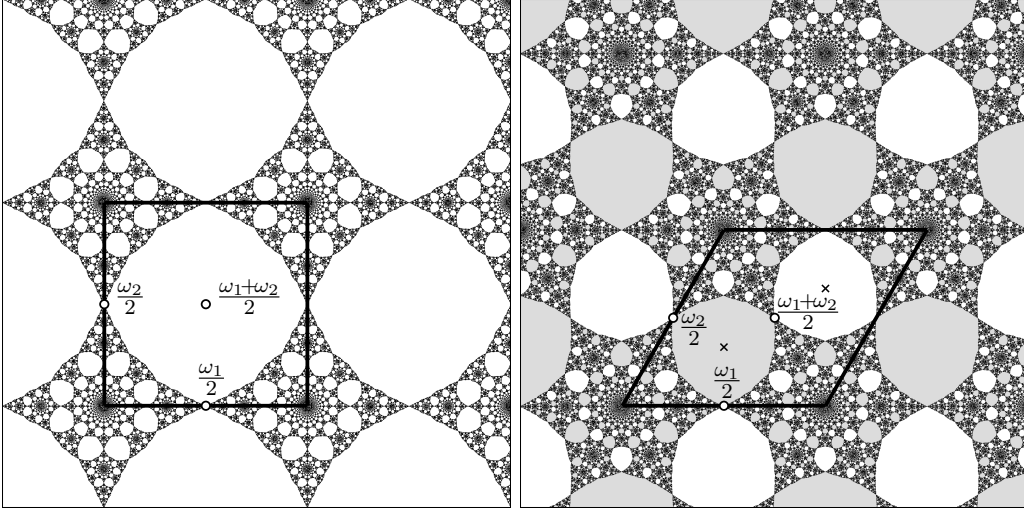


Figure 2: Julia sets corresponding to  $\tau = i$  and  $\tau = e^{i\pi/3}$ .

are marked by thick lines and the half-periods are marked by circles. For the square lattice, the half-periods are the only fixed points of  $g$ , with  $(\omega_1 + \omega_2)/2$  attracting and the two other ones repelling. For the hexagonal lattice, all half-periods are repelling and there are the attracting fixed points  $(\omega_1 + \omega_2)/3$  and  $2(\omega_1 + \omega_2)/3$  marked by crosses.

Figure 3 shows the Julia set of the function corresponding to  $\tau = 1/2 + ib_1$ . The half-period  $\omega_1/2$  is a parabolic fixed point with two petals. The two other half-periods are repelling fixed points.

*Comment.* Besides the application in [20, 18, 19, 17], anti-holomorphic dynamics were studied for its own sake in [10, 27, 26] and elsewhere. In these papers, the iteration of the anti-holomorphic map  $\bar{z}^2 + c$  was investigated. The bifurcation diagram in the  $c$ -plane for this map is called the Mandelbar

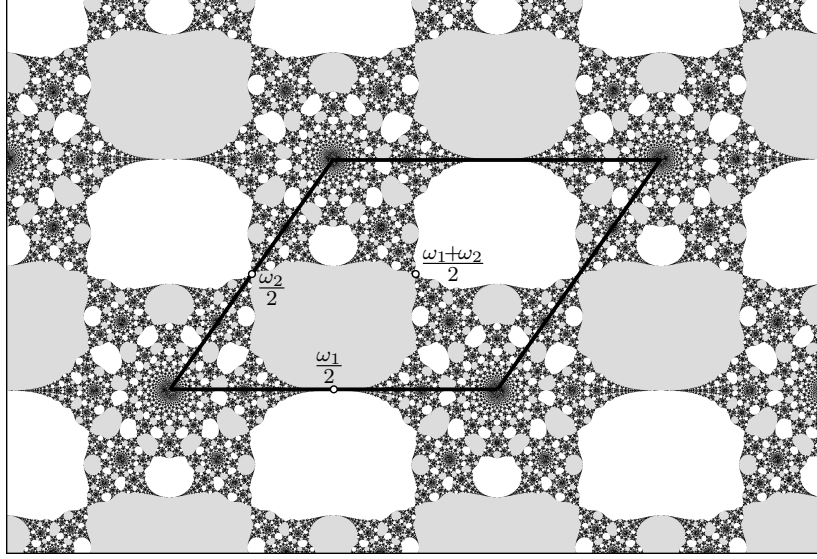


Figure 3: Julia set corresponding to  $\tau = 1/2 + ib_1$ .

set or Tricorn. It was noticed that the neutral cycle appears on a whole arc of the boundary of the hyperbolic component.

In the case we considered, a striking new phenomenon occurs: the neutral cycle occurs *everywhere* on the boundary of the hyperbolic component.

To be more precise, let  $H$  be the upper half-plane, so  $\tau \in H$ . Let  $X$  be the maximal open set of  $\tau$  where  $G$  has three critical points, and  $Y = H \setminus \overline{X}$ . For  $\tau$  in  $X$ , the critical orbits of  $g = g_\tau$  tend to an attracting fixed point, while for  $\tau \in Y$  they tend to two attracting fixed points. The open sets  $X$  and  $Y$  and their common boundary which is a piecewise analytic curve exhaust the whole parameter space! So a generic  $g_\tau$  in our family has hyperbolic dynamics in a trivial way.

Such a situation occurs in holomorphic dynamics only in trivial cases [24, Theorem 2.2], and was not encountered so far in anti-holomorphic dynamics.

The point in a hyperbolic component at which the multiplier is zero is called the center of this hyperbolic component. Computation shows that the centers of  $X$  and  $Y$  in the standard fundamental domain are  $\tau = i$  and  $\tau = e^{2\pi i/3}$ , corresponding to the square and hexagonal lattices respectively.

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