Green's function and anti-holomorphic dynamics on a torus

Walter Bergweiler and Alexandre Eremenko*

Abstract

We give a new, simple proof of the fact recently discovered by C.-S. Lin and C.-L. Wang that the Green function of a torus has either three or five critical points, depending on the modulus of the torus. The proof uses anti-holomorphic dynamics. As a byproduct we find a one-parametric family of anti-holomorphic dynamical systems for which the parameter space consists only of hyperbolic components and analytic curves separating them.

1 Introduction

Green's function on a torus T is defined as a solution of the equation

$$\Delta G = \delta - \frac{1}{|T|},$$

normalized so that

$$\int_T G = 0.$$

Here δ is the delta-function, and $|T|$ is the area of T with respect to a flat metric.

We write the torus T as $T = \mathbb{C}/\Lambda$ with a lattice

$$\Lambda = \{m\omega_1 + n\omega_2 \colon m, n \in \mathbb{Z}\},$$

*Supported by NSF grant DMS-1361836.

where $\tau = \omega_2/\omega_1$ satisfies $\operatorname{Im}\tau > 0$. Recently C.-S. Lin and C.-L. Wang [21] discovered that Green's function has either three or five critical points, depending on τ. It is surprising that this simple fact was not known until 2010. In [8, 9] they study the corresponding partition of the τ-half-plane. Their proofs are long and indirect, using advanced non-linear PDE theory and modular forms. Our paper is motivated by the desire to give a simple proof of their result that Green's function has either three or five critical points and to give a criterion for τ distinguishing which case occurs.

We have (see [21])

$$G(z) = -\frac{1}{2\pi}\log|\theta_1(z)| + \frac{(\operatorname{Im} z)^2}{2\operatorname{Im}\tau} + C(\tau),$$

where θ_1 is the first theta-function. Here and in the following we use the notation of elliptic functions as given in [1, 14]. We note that the notation in [2, 26, 27] is different, see the remark following the theorem below.

Critical points of G are solutions of the equation

$$\zeta(z) + az + b\overline{z} = 0, \tag{1}$$

where the constants a and b are uniquely defined by the condition that the left hand side is Λ-periodic. With $\zeta(z+\omega_j) = \zeta(z) + \eta_j$ for $j = 1, 2$ we thus have

$$\eta_1 + a\omega_1 + b\overline{\omega_1} = 0 \quad \text{and} \quad \eta_2 + a\omega_2 + b\overline{\omega_2} = 0.$$

With the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ we obtain

$$b = -\frac{\pi}{|\omega_1|^2\operatorname{Im}\tau} \quad \text{and} \quad a = -\frac{b\overline{\omega_1}}{\omega_1} - \frac{\eta_1}{\omega_1} = \frac{\pi}{\omega_1^2\operatorname{Im}\tau} - \frac{\eta_1}{\omega_1}. \tag{2}$$

So the problem is to determine the number of solutions of (1) where a and b are given by (2).

Theorem. *The equation* (1) *has three solutions in T if $e_j\omega_1^2 + \eta_1\omega_1 = 0$ or*

$$\operatorname{Im}\left(\frac{2\pi i}{e_j\omega_1^2 + \eta_1\omega_1} - \tau\right) \geq 0 \tag{3}$$

for some $j \in \{1, 2, 3\}$ and it has five solutions otherwise.

Here, as usual, $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$ and $e_3 = \wp((\omega_1 + \omega_2)/2)$. An elementary computation shows that the condition in the theorem is equivalent to

$$\min_{1\leq j\leq 3}\left|\frac{e_j\omega_1^2 + \eta_1\omega_1}{\pi}\operatorname{Im}\tau - 1\right| \leq 1$$

We note that $e_j\omega_1^2$ and $\eta_1\omega_1$ depend only on $\tau = \omega_2/\omega_1$. We may restrict to the case that $\omega_1 = 1$ so that $\tau = \omega_2$. Then (3) simplifies to

$$\operatorname{Im}\left(\frac{2\pi i}{e_j + \eta_1} - \tau\right) \geq 0.$$

As mentioned, a different notation for elliptic functions is used in [2, 26, 27]. There the periods are denoted by $2\omega_j$ and the definition of η_1 also differs by a factor 2. Thus in that terminology (3) takes the form

$$\operatorname{Im}\left(\frac{\pi i}{2(e_j\omega_1^2 + \eta_1\omega_1)} - \tau\right) \geq 0.$$

Figure 1 shows (in gray) the regions in the τ-plane where Green's function has 5 critical points; that is, the set of τ-values where (3) fails for all j. The range shown is $|\operatorname{Re}\tau| \leq 1$ and $0.15 \leq \operatorname{Im}\tau \leq 2.15$.

The standard fundamental domain consisting of those τ which satisfy the inequalities $\operatorname{Im}\tau > 0$, $-1/2 < \operatorname{Re}\tau \leq 1/2$ and $|\tau| \geq 1$, with $|\tau| = 1$ only if $\operatorname{Re}\tau \geq 0$, is in the upper middle of the picture. Its images under the modular group are also shown.

Our proof is based on Fatou's theorem from complex dynamics. Originally Fatou's theorem was proved to estimate from above the number of attracting cycles of a rational function. Then it was extended to more general classes of functions. The most surprising fact is that Fatou's theorem can be used sometimes to estimate the number of solutions of equations in settings where dynamics is not present. This was first noticed in [11]; the contents of this unpublished preprint is reproduced in [13, 7]. The paper [5] shows that Fatou's theorem can be used to prove under some circumstances the existence of critical points of a meromorphic function. In the papers [18, 16, 17, 15] Fatou's theorem was used to obtain upper estimates of the numbers of solutions of equations of the form

$$z = r(\overline{z}),$$

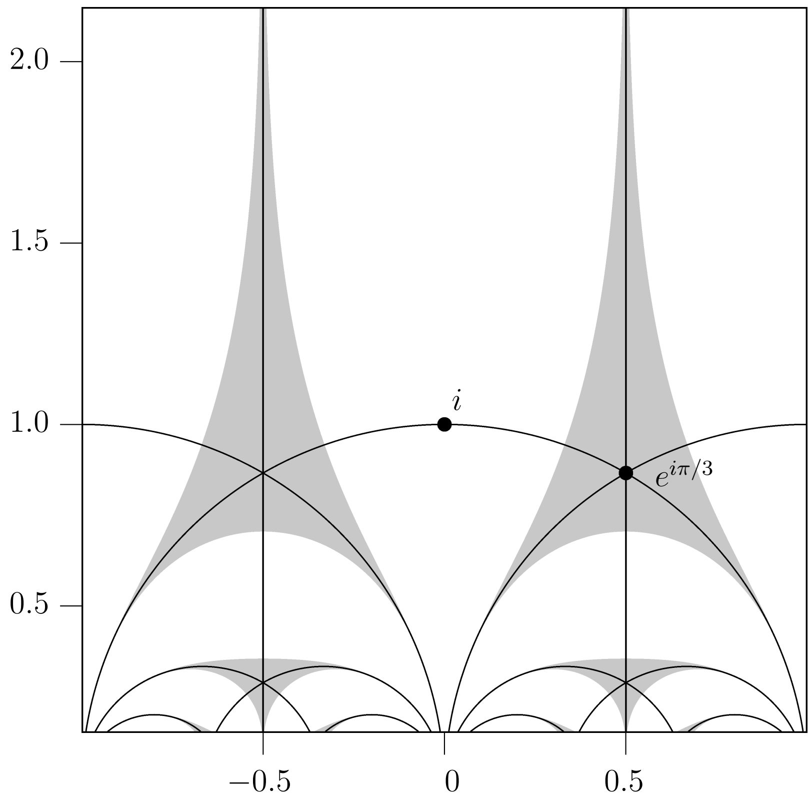

Figure 1: The regions given by (3) in the τ-plane.

with a meromorphic function r; this permitted to prove a conjecture in astronomy [16, 17]. In the recent work [19], a topological classification of quadrature domains is obtained with a method based on Fatou's theorem.

As mentioned, Fatou [12, §30] stated his result originally only for rational functions. As pointed out for example in [4, Theorem 7], the proof extends to functions meromorphic in $\mathbb{C}$. Here we need a slightly more general version, namely for functions meromorphic in $\overline{\mathbb{C}}\backslash E$, where E is a countable compact subset of $\overline{\mathbb{C}}$. For example, iterates of functions meromorphic in $\mathbb{C}$ are of this form. Again Fatou's proof extends to this context; see [3, Lemma 10], where in fact a more general class of functions is considered.

To state the version of Fatou's theorem that we need, let E be a countable compact subset of $\overline{\mathbb{C}}$ and let $f\colon \overline{\mathbb{C}}\backslash E \to \overline{\mathbb{C}}$ be meromorphic, and suppose that f does not extend meromorphically to any point of E. For simplicity we also assume that $\infty \in E$, as this will be the case in our situation.

A point $z_0 \in \mathbb{C}\backslash E$ is called *fixed* if $f(z_0) = z_0$, and for such a point $f'(z_0)$ is called the *multiplier* of z_0. A fixed point is called *attracting*, *neutral* or *repelling* depending on whether the modulus of its multiplier is less than, equal to or greater than 1, respectively. A neutral fixed point is called *rational* if the multiplier is a root of unity.

A point $c \in \mathbb{C}\backslash E$ is called a *critical* point of f if $f'(c) = 0$ and it is called an *asymptotic value* of f if there exists a curve $\gamma\colon [0,1) \to \mathbb{C}\backslash E$ such that $\gamma(t)$ tends to a point in E as $t \to 1$ while $f(\gamma(t)) \to c$ as $t \to 1$.

Fatou's Theorem. *For each attracting fixed point z_0 of f there is a critical point or an asymptotic value c such that $f^n(c) \to z_0$ as $n \to \infty$.*

Actually, Fatou's theorem also holds for rational neutral fixed points, and it also extends to periodic points, that is, fixed points of the iterates of f, but we do not need these results.

2 Proof of the Theorem

To prove the theorem we rewrite (1) as a fixed point equation

$$z = -\frac{1}{b}\left(\overline{\zeta(z)} + \overline{az}\right) =: g(z), \tag{4}$$

so g is an *anti*-meromorphic function in the plane. As the left hand side of (1) is Λ-periodic, we conclude from (4) that

$$g(z+\omega) = g(z) + \omega, \quad \omega \in \Lambda. \tag{5}$$

The function g does not map the plane into itself because it has poles, so the equation (5) does not permit to define a map of the torus T into itself. To remedy this, we consider the set P_0 of poles of g. For $n \geq 1$ we define inductively $P_n = g^{-1}(P_{n-1})$. Then all iterates of g are defined outside the closure $J(g)$ of the union $P_\infty = \bigcup_{n=0}^{\infty} P_n$. In fact, the iterates of g form a normal family in $\mathbb{C}\backslash J(g)$. Thus – with an obvious extension of these concepts from holomorphic functions to anti-holomorphic ones – we call $J(g)$ the *Julia set* and its complement $F(g) = \mathbb{C}\backslash J(g)$ the *Fatou set* of g. The Fatou set is thus the maximal open subset of the plane such that $g(F(g)) \subset F(g)$. Evidently, $F(g)$ is Λ-invariant, so the map $g\colon F(g) \to F(g)$ descends to a map which is defined on an open subset of the torus T and maps this open subset to itself.

We apply the terminology attracting, neutral and repelling also to a fixed point z_0 of g, considering $\overline{\partial} g(z_0)$ as the multiplier. Note that

$$\overline{\overline{\partial} g(z)} = -\frac{1}{b}\left(\zeta'(z) + a\right) = \frac{1}{b}(\wp(z) - a). \tag{6}$$

To obtain holomorphic dynamics instead of the anti-holomorphic one, we consider the second iterate $h = g^2$. By the chain rule, we have

$$h' = ((\overline{\partial} g) \circ g) \cdot \partial \overline{g} = ((\overline{\partial} g) \circ g) \cdot \overline{\overline{\partial} g}. \tag{7}$$

For a fixed point z_0 of g we thus obtain $h'(z_0) = |\overline{\partial} g(z_0)|^2$. Even though we will not need this fact, we observe that the multiplier with respect to h at a fixed point of g is always a non-negative real number.

In order to apply Fatou's theorem, we have to consider the critical points and asymptotic values of h.

Lemma 1. *The map $h\colon \mathbb{C}\backslash P_\infty \to \mathbb{C}$ has no asymptotic values.*

Proof. From (5) we see that $g(z) \to \infty$ as $z \to \infty$ avoiding small neighborhoods of the poles. So g cannot have an asymptotic value in $\mathbb{C}$, and its second iterate h also cannot. $\square$

We consider the equivalence relation defined by $z \sim z'$ if $h^n(z) = h^m(z')$ for some non-negative integers m and n. The equivalence classes are called the *grand orbits*. We note that if the sequence $(h^n(z))$ converges for some $z \in T$, then for all z' in the grand orbit of z the sequence $(h^n(z'))$ converges to the same limit. We call this the limit of the grand orbit.

Lemma 2. *The set of critical points of h belongs to at most 4 grand orbits under h.*

Proof. By (6), the equation $\overline{\partial} g(z) = 0$ is equivalent to $\wp(z) = a$, and has two solutions which we denote c and $-c$. By (7) the other zeros of h' are $g^{-1}(\pm c) = \pm g^{-1}(c)$. Even though this is an infinite set on T, the critical points of h' are thus contained in at most 4 grand orbits represented by c, $-c$, $g(c)$ and $-g(c)$. $\square$

Lemma 3. *Let z_0 be a fixed point of g. If $h^n(z) \to z_0$ as $n \to \infty$ for some z, then $h^n(g(z)) \to z_0$.*

Proof. We have $h^n \circ g = g \circ h^n$ for all $n \in \mathbb{N}$. Since $h^n(z) \to z_0$ this yields $h^n(g(z_0)) = g(h^n(z)) \to g(z_0) = z_0$. $\square$

Using Fatou's theorem, we deduce the following result from the previous three lemmas.

Lemma 4. *The function g has at most two attracting fixed points.*

The map $\phi\colon T \to \overline{\mathbb{C}}$, $z \mapsto z - g(z)$, is well defined by (5). Let $J_\phi = 1 - |\overline{\partial} g|^2$ be the Jacobian determinant of ϕ. Then $D^+ = \{z \in T\colon J_\phi(z) > 0\}$ is the set where ϕ preserves the orientation and $D^- = \{z \in T\colon J_\phi(z) < 0\}$ is the set where ϕ reverses the orientation. As ϕ has one pole, a point w of large modulus has one preimage on T and the map is reversing orientation at this preimage. We conclude that the degree of ϕ equals -1; see [23, §5] for the definition of the degree.

Suppose first that all zeros of ϕ are in $D^+ \cup D^-$. Equivalently, g has no neutral fixed points. Denote by N^+ and N^- the numbers of zeros of ϕ in D^+ and D^- respectively. Then $N^+ - N^-$ equals the degree of ϕ so that $N^+ - N^- = -1$ and thus

$$N^- = N^+ + 1. \tag{8}$$

For the number $N = N^+ + N^-$ of fixed points of g in T we thus find that

$$N = 2N^+ + 1. \tag{9}$$

Since $J_\phi = 1 - |\overline{\partial} g|^2$, the zeros of ϕ in D^+ are attracting fixed points of g while the zeros of ϕ in D^- are repelling fixed points of g. Thus Lemma 4 yields that

$$N^+ \leq 2. \tag{10}$$

It follows from (9) and (10) that $N \leq 5$.

On the other hand, since ϕ is odd and Λ-periodic it easily follows that that the half-periods $\omega_1/2$, $\omega_2/2$ and $\omega_3/2 = (\omega_1 + \omega_2)/2$ are zeros of ϕ. Equivalently, they are fixed points of g. Thus we have $N \geq 3$. Altogether, since N is odd by (9), it follows that $N = 3$ or $N = 5$.

It remains to determine the criterion distinguishing the cases. Suppose first that all three half-periods are in D^-; that is, they are repelling fixed points of g. Then $N^- \geq 3$. This yields that $N^- = 3$ and $N^+ = 2$ so that $N = 5$ by (8) and (10). Suppose now that one half-period, say ω_j, is not in D^- and thus in D^+. Thus ω_j is an attracting fixed point of g and hence attracts a critical orbit by Fatou's theorem. However, since g is odd we see that ω_j in fact attracts both critical orbits. Thus, by Fatou's theorem, there are no other attracting fixed points. Thus $N^+ = 1$ and hence $N = 3$ in this case.

We see that $N = 3$ if and only if there exists $j \in \{1, 2, 3\}$ such that ω_j is an attracting fixed point of g; that is, $|\overline{\partial} g(\omega_j)| < 1$. Using (6) this takes the form

$$\min_{1\le j\le 3} \left| \frac{e_j}{b} - \frac{a}{b} \right| < 1. \tag{11}$$

Now (2) yields

$$\frac{a}{b} = -\frac{\overline{\omega_1}}{\omega_1} - \frac{\eta_1}{b\omega_1} = -\frac{\overline{\omega_1}}{\omega_1} + \frac{\eta_1 |\omega_1|^2 \operatorname{Im}\tau}{\omega_1 \pi} = \frac{\overline{\omega_1}}{\omega_1} \left(-1 + \frac{\eta_1 \omega_1 \operatorname{Im}\tau}{\pi} \right)$$

and

$$\frac{e_j}{b} = -\frac{e_j |\omega_1|^2 \operatorname{Im}\tau}{\pi} = -\frac{\overline{\omega_1}}{\omega_1} \frac{e_j \omega_1^2 \operatorname{Im}\tau}{\pi}.$$

Substituting the last two equations in (11) yields

$$\left| \frac{e_j \omega_1^2 + \eta_1 \omega_1}{\pi} \operatorname{Im}\tau - 1 \right| < 1.$$

This is equivalent to strict inequality in (3). This completes the proof of the theorem in the case that all zeros of ϕ are in $D^+ \cup D^-$.

To deal with the case where this condition is not satisfied, we note that in the above arguments we may replace $\phi(z)$ by $\psi(z) = \phi(z) - w$ for any $w \in \mathbb{C}$. Noting that ϕ and ψ have the same Jacobian determinant we conclude that whenever all w-points of ϕ are in $D^+ \cup D^-$, then ϕ has either three or five w-points in T. Moreover, there are three w-points if and only if one them is in D^+.

We will use the following result [6, Proposition 3].

Lemma 5. *Let $D \subset \mathbb{C}$ be a domain and let $f\colon D \to \mathbb{C}$ be harmonic. Suppose there exists $m \in \mathbb{N}$ such that every $w \in \mathbb{C}$ has at most m preimages. Then the set of points which have m preimages is open.*

Suppose now that ϕ has a zero in $T \backslash (D^+ \cup D^-)$. Then there are arbitrarily small w such that ϕ has a w-point in D^+. Since for such w the function ϕ has three w-points, it follows from Lemma 5 that ϕ has three zeros in T. These zeros are the half-periods and we see that we have equality in (3). This completes the proof of the theorem.

Remark. The quantities $e_j\omega_1^2+\eta_1\omega_1$ occurring in (3) have the following representations via theta functions, see [26, p. 44]:

$$e_1\omega_1^2+\eta_1\omega_1=-\frac{\vartheta_2''(0)}{\vartheta_2(0)},\quad e_2\omega_1^2+\eta_1\omega_1=-\frac{\vartheta_0''(0)}{\vartheta_0(0)},\quad e_3\omega_1^2+\eta_1\omega_1=-\frac{\vartheta_3''(0)}{\vartheta_3(0)}.$$

The series for theta functions converge fast and provide a convenient way to compute Figure 1.

Let

$$F_j(\tau)=\frac{2\pi i}{e_j\omega_1^2+\eta_1\omega_1}-\tau$$

so that (3) takes the form $\operatorname{Im} F_j(\tau)\geq 0$. It seems that F_j maps the components of the set of all τ in the upper half-plane where $\operatorname{Im} F_j(\tau)>0$ univalently onto the upper half-plane, but we have not been able to prove this.

Comment. Besides the application in [18, 16, 17, 15], anti-holomorphic dynamics were studied for its own sake in [10, 25, 24] and elsewhere. In these papers, the iteration of the anti-holomorphic map $\overline{z}^2+c$ was investigated. The bifurcation diagram in the c-plane for this map is called the Mandelbar set or Tricorn. It was noticed that the neutral cycle appears on a whole arc of the boundary of the hyperbolic component.

In the case we considered, a striking new phenomenon occurs: the neutral cycle occurs *everywhere* on the boundary of the hyperbolic component.

To be more precise, let H be the upper half-plane, so $\tau\in H$. Let X be the maximal open set of τ where G has three critical points, and $Y=H\backslash\overline{X}$. For τ in X, the critical orbits of $g=g_\tau$ tend to an attracting fixed point, while for $\tau\in Y$ they tend to two attracting fixed points. The open sets X and Y and their common boundary which is a piecewise analytic curve exhaust the whole parameter space! So a generic g_τ in our family has hyperbolic dynamics in a trivial way.

Such a situation occurs in holomorphic dynamics only in trivial cases [22, Theorem 2.2], and was not encountered so far in anti-holomorphic dynamics.

The point in a hyperbolic component at which the multiplier is zero is called the center of this hyperbolic component. Computation shows that the centers of X and Y in the standard fundamental domain are $\tau=i$ and $\tau=e^{2\pi i/3}$, corresponding to the square and hexagonal lattices respectively.

References

[1] L. Ahlfors, Complex analysis. An introduction to the theory of analytic functions of one complex variable. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[2] N. I. Akhiezer, Elements of the theory of elliptic functions, Amer. Math. Soc., Providence, RI, 1990.

[3] I. N. Baker, P. Domínguez and M. E. Herring, Dynamics of functions meromorphic outside a small set. Ergodic Theory Dynam. Systems 21 (2001), no. 3, 647–672.

[4] W. Bergweiler, Iteration of meromorphic functions. Bull. Amer. Math. Soc. 29 (1993), no. 2, 151–188.

[5] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoamericana 11 (1995), no. 2, 355–373.

[6] W. Bergweiler and A. Eremenko, On the number of solutions of a transcendental equation arising in the theory of gravitational lensing, Comput. Methods Funct. Theory 10 (2010), no. 1, 303–324

[7] W. Bergweiler, A. Eremenko and J. Langley, Zeros of differential polynomials in real meromorphic functions, Proc. Edinb. Math. Soc. 48 (2005), no. 2, 279–293.

[8] C.-L. Chai, C.-S. Lin and C.-L. Wang, Mean field equations, hyperelliptic curves and modular forms: I, Camb. J. Math. 3 (2015), no. 1–2, 127–274.

[9] C.-L. Chai, C.-S. Lin and C.-L. Wang, Mean field equations, hyperelliptic curves and modular forms: II, arXiv: 1502.03295.

[10] W. D. Crowe, R. Hasson, P. J. Rippon and P. E. Strain-Clark, On the structure of the Mandelbar set, Nonlinearity 2 (1989), no. 4, 541–553.

[11] A. Eremenko, Distribution of zeros of some real polynomials and iteration theory, preprint, Inst. for Low Temperature Physics and Engineering, Kharkov, 1989.

[12] P. Fatou, Sur les équations fonctionelles (deuxième mémoire), Bull. Soc. Math. France 48 (1920), 33–94.

[13] A. Hinkkanen, Iteration and the zeros of the second derivative of a meromorphic function, Proc. London Math. Soc. 65 (1992), no. 3, 629–650.

[14] A. Hurwitz, Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen, Springer-Verlag, Berlin-New York 1964.

[15] D. Khavinson and E. Lundberg, Transcendental harmonic mappings and gravitational lensing by isothermal galaxies. Complex Anal. Oper. Theory 4 (2010), no. 3, 515–524.

[16] D. Khavinson and G. Neumann, On the number of zeros of certain rational harmonic functions, Proc. Amer. Math. Soc. 134 (2006), no. 4, 1077–1085.

[17] D. Khavinson and G. Neumann, From the fundamental theorem of algebra to astrophysics: a "harmonious" path, Notices Amer. Math. Soc. 55 (2008), no. 6, 666–675.

[18] D. Khavinson and G. Świątek, On the number of zeros of certain harmonic polynomials, Proc. Amer. Math. Soc. 131 (2003), no. 2, 409–414.

[19] S.-Y. Lee and N. G. Makarov, Topology of quadrature domains, J. Amer. Math. Soc., electronically published on May 11, 2015, DOI: http://dx.doi.org/10.1090/jams828 (to appear in print).

[20] S.-Y. Lee and N. G. Makarov, Sharpness of connectivity bounds for quadrature domains, arXiv: 1411.3415.

[21] C.-S. Lin and C.-L. Wang, Elliptic functions, Green functions and the mean field equations on tori, Ann. of Math. (2) 172 (2010), no. 2, 911–954.

[22] C. McMullen, Families of rational maps and iterative root-finding algorithms, Ann. of Math. (2) 125 (1987), no. 3, 467–493.

[23] J. W. Milnor, Topology from the differentiable viewpoint. Based on notes by David W. Weaver. Revised reprint of the 1965 original. Princeton University Press, Princeton, NJ, 1997.

[24] S. Mukherjee, S. Nakane and D. Schleicher, On multicorns and unicorns II: bifurcations in spaces of antiholomorphic polynomials, arXiv: 1404.5031.

[25] S. Nakane and D. Schleicher, On multicorns and unicorns. I. Antiholomorphic dynamics, hyperbolic components and real cubic polynomials, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (2003), no. 10, 2825–2844.

[26] K. Weierstrass, Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen, edited by H. A. Schwarz, Springer, Berlin, 1893.

[27] E. T. Whittaker and G. N. Watson, A course of modern analysis, 4th edition, Cambridge University Press, Cambridge, 1927.

Mathematatisches Seminar
Christian-Albrechts-Universiat zu Kiel
Ludewig-Meyn-Str. 4
24098 Kiel
Germany

Purdue University
West Lafayette, IN 47907
USA