Radially distributed values and normal families, II

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Dedicated to Larry Zalcman

Abstract

We consider the family of all functions holomorphic in the unit disk for which the zeros lie on one ray while the 1-points lie on two different rays. We prove that for certain configurations of the rays this family is normal outside the origin.

1 Introduction and results

There is an extensive literature on entire functions whose zeros and 1-points are distributed on finitely many rays. One of the first results of this type is the following theorem of Biernacki [5, p. 533] and Milloux [11].

Theorem A. *There is no transcendental entire function for which all zeros lie on one ray and all* 1*-points lie on a different ray.*

Biernacki and Milloux proved this under the additional hypothesis that the function considered has finite order, but by a later result of Edrei [6] this is always the case if all zeros and 1-points lie on finitely many rays.

A thorough discussion of the cases in which an entire function can have its zeros on one system of rays and its 1-points on another system of rays, intersecting the first one only at 0, was given in [4]. Special attention was paid to the case where the zeros are on one ray while the 1-points are on two rays. For this case the following result was obtained [4, Theorem 2].

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Theorem B. *Let f be a transcendental entire function whose zeros lie on a ray L_0 and whose 1-points lie on two rays L_1 and L_{-1}, each of which is distinct from L_0. Suppose that the numbers of zeros and 1-points are infinite. Then $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$.*

The hypothesis that f has infinitely many zeros excludes the example $f(z) = e^z$ in which case we have $\angle(L_1, L_{-1}) = \pi$, and L_0 can be taken arbitrarily. Without this hypothesis we have the following result.

Theorem B′. *Let f be a transcendental entire function whose zeros lie on a ray L_0 and whose 1-points lie on two rays L_1 and L_{-1}, each of which is distinct from L_0. Then $\angle(L_1, L_{-1}) = \pi$ or $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$.*

Bloch's heuristic principle says that the family of all functions holomorphic in some domain which have a certain property is likely to be normal if there does not exist a non-constant entire function with this property. More generally, properties which are satisfied only by "few" entire functions often lead to normality. We refer to [2], [14] and [16] for a thorough discussion of Bloch's principle.

The following normal family analogue of Theorem A was proved in [3, Theorem 1.1]. Here $\mathbb{D}$ denotes the unit disk.

Theorem C. *Let L_0 and L_1 be two distinct rays emanating from the origin and let $\mathcal{F}$ be the family of all functions holomorphic in $\mathbb{D}$ for which all zeros lie on L_0 and all 1-points lie on L_1. Then $\mathcal{F}$ is normal in $\mathbb{D}\backslash\{0\}$.*

The purpose of this paper is to prove a normal family analogue of Theorem B′.

Theorem 1.1. *Let L_0, L_1 and L_{-1} be three distinct rays emanating from the origin and let $\mathcal{F}$ be the family of all functions holomorphic in $\mathbb{D}$ for which all zeros lie on L_0 and all 1-points lie on $L_1 \cup L_{-1}$. Assume that neither $\angle(L_{-1}, L_1) = \pi$ nor $\angle(L_0, L_1) = \angle(L_0, L_{-1}) < \pi/2$. Then $\mathcal{F}$ is normal in $\mathbb{D} \setminus \{0\}$.*

It was shown in [4, Theorem 3] that if α is of the form $\alpha = 2\pi/n$ with $n \in \mathbb{N}$, $n \geq 5$, then there exist rays L_0 and $L_{\pm 1}$ with $\angle(L_0, L_1) = \angle(L_0, L_{-1}) = \alpha$ and an entire function f with all zeros on L_0 and all 1-points on L_1 and L_{-1}. In [7] such an entire function f was constructed for every $\alpha \in (0, \pi/3]$.

The functions constructed in [4, 7] have the property that $f(re^{i\theta}) \to 0$ as $r \to \infty$ for $|\theta| < \alpha$ while $f(re^{i\theta}) \to \infty$ as $r \to \infty$ for $\alpha < |\theta| \leq \pi$. Considering the family $\{f(kz)\}_{k\in\mathbb{N}}$ we see that the conclusion of Theorem 1.1 does not hold if $\angle(L_0, L_1) = \angle(L_0, L_{-1}) \in (0, \pi/3] \cup \{2\pi/5\}$. The example $\{e^{kz}\}_{k\in\mathbb{N}}$ shows that it does not hold if $\angle(L_{-1}, L_1) = \pi$.

The question whether the conclusion of Theorem 1.1 holds if $\angle(L_0, L_1) = \angle(L_0, L_{-1}) \in (\pi/3, \pi/2) \setminus \{2\pi/5\}$ remains open.

We note that Theorem B′ follows from Theorem 1.1. To see this we only have to note that if f is a transcendental entire function and (z_k) is a sequence tending to ∞ such that $|f(z_k)| \leq 1$ for all $k \in \mathbb{N}$, then $\{f(2|z_k|z)\}_{k\in\mathbb{N}}$ is not normal at some point of modulus $\frac{1}{2}$; see the remark after Theorem 1.1 in [4].

A key tool in the theory of normal families is Zalcman's lemma [15]; see Lemma 2.1 below. An extension of this result (Lemma 2.2 below) was also crucial in the proof of Theorem C in [3]. In fact, this extension was used to prove the following result [3, Theorem 1.3] from which Theorem C can be deduced.

Theorem D. *Let D be a domain and let L be a straight line which divides D into two subdomains D^+ and D^-. Let $\mathcal{F}$ be a family of functions holomorphic in D which do not have zeros in D and for which all 1-points lie on L.*

Suppose that $\mathcal{F}$ is not normal at $z_0 \in D \cap L$ and let (f_k) be a sequence in $\mathcal{F}$ which does not have a subsequence converging in any neighborhood of z_0. Suppose that $(f_k|_{D^+})$ converges. Then either $f_k|_{D^+} \to 0$ and $f_k|_{D^-} \to \infty$ or $f_k|_{D^+} \to \infty$ and $f_k|_{D^-} \to 0$.

Note that $\mathcal{F}$ is normal in D^+ by Montel's theorem. So it is no restriction to assume that $(f_k|_{D^+})$ converges, since this can be achieved by passing to a subsequence.

Theorem D will also play an important role in the proof of Theorem 1.1. However, we will also need the following addendum to Theorem D. Here and in the following $D(a, r)$ and $\overline{D}(a, r)$ denote the open and closed disk of radius r centered at a point $a \in \mathbb{C}$.

Proposition 1.1. *Let D, L, $\mathcal{F}$, z_0 and (f_k) be as in Theorem D. Let $r > 0$ with $\overline{D}(z_0, r) \subset D$. Then for sufficiently large k there exists a 1-point a_k of f_k such that $a_k \to z_0$ and if M_k is the line orthogonal to L which intersects L at a_k, then $|f_k(z)| \neq 1$ for $z \in M_k \cap \overline{D}(z_0, r) \setminus \{a_k\}$.*

For large k this yields that $|f_k(z)| > 1$ for $z \in M_k \cap D^+ \cap \overline{D}(z_0, r)$ and $|f_k(z)| < 1$ for $z \in M_k \cap D^- \cap \overline{D}(z_0, r)$, or vice versa.

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2 Preliminaries

The lemma of Zalcman already mentioned in the introduction is the following.

Lemma 2.1. (Zalcman's Lemma) *Let $\mathcal{F}$ be a family of functions meromorphic in a domain D in $\mathbb{C}$. Then $\mathcal{F}$ is not normal at a point $z_0 \in D$ if and only if there exist*

(i) points $z_k \in D$ with $z_k \to z_0$,

(ii) positive numbers ρ_k with $\rho_k \to 0$,

(iii) functions $f_k \in \mathcal{F}$

such that

$$f_k(z_k + \varrho_k z) \to g(z)$$

locally uniformly in $\mathbb{C}$ with respect to the spherical metric, where g is a non-constant meromorphic function in $\mathbb{C}$.

In the proof (see also [1, Section 4] or [16, p. 217f] besides [15]) one considers the spherical derivative

$$g_k^{\#}(z) = \frac{|g_k'(z)|}{1 + |g_k(z)|^2}$$

of the function g_k defined by

$$g_k(z) = f_k(z_k + \varrho_k z) \tag{2.1}$$

and shows that for suitably chosen f_k, z_k, ϱ_k and R_k with $R_k \to \infty$ we have $g_k^{\#}(0) = 1$ as well as

$$g_k^{\#}(z) \le 1 + o(1) \quad \text{for } |z| \le R_k \text{ as } k \to \infty.$$

Marty's theorem then implies that (g_k) has a locally convergent subsequence.

The following addendum to Lemma 2.1 was proved in [3, Lemma 2.2].

Lemma 2.2. *Let $t_0 > 0$ and $\varphi\colon [t_0, \infty) \to (0, \infty)$ be a non-decreasing function such that $\varphi(t)/t \to 0$ as $t \to \infty$ and*

$$\int_{t_0}^{\infty} \frac{dt}{t\varphi(t)} < \infty.$$

Then one may choose z_k, ϱ_k and f_k in Zalcman's Lemma 2.1 such that

$$R_k := \frac{1}{\varrho_k \varphi(1/\varrho_k)} \to \infty \quad \text{as } k \to \infty.$$

and the functions g_k given by (2.1) are defined in the disks $D(0, R_k)$ and satisfy

$$g_k^{\#}(z) \le 1 + \frac{|z|}{R_k} \quad \text{for } |z| < R_k. \tag{2.2}$$

The next lemma is standard [12, Proposition 1.10].

Lemma 2.3. *Let Ω be a convex domain and let $f\colon \Omega \to \mathbb{C}$ be holomorphic. If $\operatorname{Re} f'(z) > 0$ for all $z \in \Omega$, then f is univalent.*

The following result can be found in [8, p. 112].

Lemma 2.4. *Let $a \in \mathbb{C}$, $r > 0$ and let $f\colon D(a, r) \to \mathbb{C}$ be univalent. Then*

$$\left| \arg\left(\frac{f(z) - f(a)}{f'(a)(z-a)} \right) \right| \le \log \frac{1 + \dfrac{|z-a|}{r}}{1 - \dfrac{|z-a|}{r}}$$

for $z \in D(a, r)$.

The result is stated in [8] for the special case that $a = 0$, $r = 1$, $f(0) = 0$ and $f'(0) = 1$, but the version given above follows directly from this special case.

Proof of Proposition 1.1. We recall some arguments of the proof of Theorem D in [3] and then describe the additional arguments that have to be made.

As in [3] we may assume that $L = \mathbb{R}$ and we use Zalcman's Lemma 2.1 as well as Lemma 2.2, applied with $\varphi(t) = (\log t)^2$, to obtain a sequence (z_k) tending to z_0 and a sequence (ρ_k) tending to 0 such that

$$R_k := \frac{1}{\varrho_k \left(\log \varrho_k\right)^2} \to \infty,$$

and the function g_k given by (2.1) is defined in the disk $D(0, R_k)$ and satisfies (2.2) and $g_k^{\#}(0) = 1$.

As in [3, Proof of Theorem 1.3] we find a sequence (b_k) of 1-points of g such that

$$g_k(z) = \exp(c_k(z-b_k) + \delta_k(z)),$$

where (see [3, (3.4) and (3.5)])

$$|c_k + 2i| \le \frac{C}{R_k} \quad \text{or} \quad |c_k - 2i| \le \frac{C}{R_k} \tag{2.3}$$

with some constant C and (see [3, (2.22)])

$$|\delta_k(z)| \le 2^7 \frac{|z-b_k|^2}{R_k} \quad \text{for } |z-b_k| \le \frac{1}{16} R_k. \tag{2.4}$$

Without loss of generality we may assume that the first alternative holds in (2.3).

We put

$$h_k(z) = c_k(z-b_k) + \delta_k(z)$$

so that $g_k(z) = \exp h_k(z)$. We will show that h_k is univalent in $D(b_k, 2s_k)$ where $s_k = 2^{-11} R_k$. In order to do so we note that for $|z-b_k| \le 2s_k$ we have

$$|\delta_k'(z)| = \frac{1}{2\pi} \left| \int_{|\zeta-b_k|=4s_k} \frac{\delta_k(\zeta)}{(z-\zeta)^2} d\zeta \right| \le 4s_k \frac{1}{(2s_k)^2} \max_{|\zeta-b_k|=4s_k} |\delta_k(\zeta)|.$$

Since $4s_k = 2^{-9} R_k < R_k/16$ we may apply (2.4) to estimate the maximum on the right hand side and obtain

$$|\delta_k'(z)| \le \frac{1}{s_k} 2^7 \frac{(4s_k)^2}{R_k} = 1 \quad \text{for } |z-b_k| \le 2s_k.$$

Thus, since we assumed that the first alternative holds in (2.3),

$$\mathrm{Re}(ih_k'(z)) = \mathrm{Re}(ic_k + i\delta_k'(z)) \ge 2 - \frac{C}{R_k} - 1 > 0$$

for $z \in D(b_k, 2s_k)$ if k is sufficiently large. Lemma 2.3 implies that ih_k and hence h_k are univalent in this disk. Since $h_k(b_k) = 0$ and, by (2.4), $\delta_k'(b_k) = 0$ and thus $h_k'(b_k) = c_k$, Lemma 2.4 now yields that if $z \in \overline{D}(b_k, s_k)$, then

$$\left| \arg\left(\frac{h_k(z)}{c_k(z-b_k)} \right) \right| \le \log 3.$$

For $t \in \mathbb{R}$ with $0 < |t| \le s_k$ we thus have

$$\left| \arg\left(\frac{h_k(b_k + it)}{ic_k t} \right) \right| \le \log 3.$$

Since we assumed that the first alternative holds in (2.3), this implies for large k that

$$|\arg(h_k(b_k + it))| \le \log 3 + \arcsin\left(\frac{C}{2R_k} \right) < \frac{1}{2}\pi \quad \text{for } 0 < t \le s_k$$

while

$$|\arg(h_k(b_k + it)) - \pi| < \frac{1}{2}\pi \quad \text{for } -s_k \le t < 0.$$

Hence

$$\operatorname{Re}(h_k(b_k + it)) \begin{cases} > 0 & \text{if } 0 < t \le s_k, \\ < 0 & \text{if } -s_k \le t < 0, \end{cases}$$

so that

$$|g_k(b_k + it)| = \exp(\operatorname{Re}(h_k(b_k + it))) \begin{cases} > 1 & \text{if } 0 < t \le s_k, \\ < 1 & \text{if } -s_k \le t < 0. \end{cases} \tag{2.5}$$

As in [3, (3.2), (3.6) and (3.7)] we put

$$a_k = z_k + \rho_k b_k, \quad u_k = b_k + is_k = b_k + i2^{-11}R_k \quad \text{and} \quad \alpha_k = z_k + \rho_k u_k.$$

By (2.1) and (2.5) we have $|f_k(z)| > 1$ for z in the line segment $(a_k, \alpha_k]$. Choose $d > r$ such that $\overline{D}(z_0, d) \in D$. We put $\beta_k = z_k + id$. Then $\beta_k \in D^+$ for large k and as in [3] we can use Landau's theorem to show that we also have $|f_k(z)| > 1$ for $z \in [\alpha_k, \beta_k]$. Altogether thus $|f_k(z)| > 1$ for $z \in (a_k, \beta_k]$ and hence for $z \in M_k \cap D^+ \cap \overline{D}(z_0, r)$ and large k. Analogously, $|f_k(z)| < 1$ for $z \in M_k \cap D^- \cap \overline{D}(z_0, r)$ and large k. □

Lemma 2.5. *Let $0 < \alpha < \pi$ and $\alpha < \beta < 2\pi - \alpha$. Let $u\colon \mathbb{D} \to [-\infty, \infty)$ be a subharmonic function which is harmonic in $\mathbb{D} \setminus \{re^{i\beta} : 0 \le r < 1\}$. Suppose that $u(z) > 0$ for $|\arg z| < \alpha$ while $u(z) \le 0$ for $\alpha \le |\arg z| \le \pi$. Then $\alpha \ge \pi/2$. Moreover, if $\alpha > \pi/2$, then $\beta = \pi$. In addition, if u is harmonic in $\mathbb{D} \setminus \{0\}$, then $\alpha = \pi/2$.*

Proof. Let $\gamma = 2\alpha/\pi$ and define $v\colon \{z \in \mathbb{D}\colon \operatorname{Re} z \geq 0\} \to [-\infty, \infty)$ by $v(z) = u(z^\gamma)$. Then $v(z) > 0$ for $\operatorname{Re} z > 0$ while $v(z) \leq 0$ for $\operatorname{Re} z = 0$. In fact, $v(z) = 0$ for $\operatorname{Re} z = 0$ by upper semicontinuity. We have $v = \operatorname{Re} f$ for some function f holomorphic in $\{z \in \mathbb{D}\colon \operatorname{Re} z > 0\}$. By the Schwarz reflection principle f extends to a function holomorphic in $\mathbb{D}$. Hence f has a power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ convergent in $\mathbb{D}$. With $\delta = 1/\gamma$ we thus have

$$u(z) = v(z^{1/\gamma}) = \operatorname{Re} f(z^\delta) = \operatorname{Re}\left(\sum_{k=0}^{\infty} a_k z^{k\delta}\right)$$

for $z \in \mathbb{D} \setminus \{re^{i\beta}\colon 0 \leq r < 1\}$, meaning that

$$u(re^{i\theta}) = \operatorname{Re}\left(\sum_{k=0}^{\infty} a_k r^\delta e^{ik\delta\theta}\right)$$

for $0 < r < 1$ and $\beta - 2\pi < \theta < \beta$.

Since $\operatorname{Re} f(z) = v(z) > 0$ for $\operatorname{Re} z > 0$ and $\operatorname{Re} f(z) = 0$ for $\operatorname{Re} z = 0$ we find that $\operatorname{Re} a_0 = 0$ and $a_1 > 0$. It follows that

$$u(re^{i\theta}) = a_1 r^\delta \cos(\delta\theta) + \mathcal{O}(r^{2\delta})$$

as $r \to 0$, uniformly for $\beta - 2\pi < \theta < \beta$. We may assume that $\beta \geq \pi$. The condition that $u(re^{i\theta}) \leq 0$ for $\alpha \leq \theta < \beta$ then implies that $\delta\pi \leq \delta\beta \leq 3\pi/2$ so that $\delta \leq 3/2$. Suppose that $\delta \neq 1$. Since u is subharmonic and a connected set containing more than one point is non-thin at every point of its closure [13, Theorem 3.8.3], we have $u(re^{i\beta}) = u(re^{i(\beta-2\pi)})$ and thus $\cos(\delta\beta) = \cos(\delta(\beta - 2\pi))$. This yields that $\beta = \pi$. Since u is subharmonic, we also have

$$\begin{aligned} 0 = u(0) &\leq \frac{1}{2\pi} \int_{\beta-2\pi}^{\beta} u(re^{i\theta}) d\theta \\ &= a_1 r^\delta \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\delta\theta) d\theta + \mathcal{O}(r^{2\delta}) \\ &= a_1 r^\delta \frac{1}{\delta\pi} \sin(\delta\pi) + \mathcal{O}(r^{2\delta}). \end{aligned}$$

Hence $\sin(\delta\pi) \geq 0$. Since $\delta \leq 3/2$ and since we assumed that $\delta \neq 1$ this implies that $\delta < 1$. Overall thus $\delta \leq 1$ so that $\alpha = \gamma\pi/2 = \pi/(2\delta) \geq \pi/2$, and if $\alpha > \pi/2$ so that $\delta < 1$, then $\beta = \pi$. Finally, u can be harmonic only if $\delta = 1$, which means that $\alpha = \pi/2$. □

For a bounded domain G, a point $z \in G$ and a compact subset A of ∂G let $\omega(z, A, G)$ denote the harmonic measure of A at a point $z \in G$; see, e.g., [13, §4.3]. It is the solution of the Dirichlet problem for the characteristic function χ_A of A on the boundary of G. Thus

$$\omega(z, A, G) = \sup_u u(z), \tag{2.6}$$

where the supremum is taken over all functions u subharmonic in G which satisfy $\limsup_{z \to \zeta} u(z) \le \chi_A(\zeta)$ for all $\zeta \in \partial G$.

Lemma 2.6. *Let G and H be bounded domains and let $A \subset \partial G$ and $B \subset \partial H$ be compact. If $G \subset H$ and $A \supset \partial G \cap (H \cup B)$, then $\omega(z, A, G) \ge \omega(z, B, H)$ for all $z \in G$.*

Proof. Let $\zeta \in \partial G \setminus A$. Then $\zeta \in \partial G \setminus (H \cup B)$ and thus $\zeta \in \partial H \setminus B$. Hence $\lim_{z \to \zeta} \omega(z, B, H) = 0$. We conclude that $\limsup_{z \to \zeta} \omega(z, B, H) \le \chi_A(\zeta)$ for all $\zeta \in \partial G$. Since $u(z) = \omega(z, B, H)$ is an admissible choice in (2.6), the conclusion follows. □

3 Proof of Theorem 1.1

Without loss of generality we may assume that L_1 and L_{-1} are symmetric with respect to the real axis and that L_1 is in the upper half-plane. Thus $L_{\pm 1} = \{re^{\pm i\alpha} \colon r \ge 0\}$ for some $\alpha \in (0, \pi)$. We may also assume that $L_0 = \{re^{i\beta} \colon r \ge 0\}$ where $\alpha < \beta < 2\pi - \alpha$. We define

$$\begin{aligned} S &= \{re^{i\theta} \colon 0 < r < 1,\ |\theta| < \alpha\}, \\ S^+ &= \{re^{i\theta} \colon 0 < r < 1,\ \alpha < \theta < \beta\}, \\ S^- &= \{re^{i\theta} \colon 0 < r < 1,\ \beta < \theta < 2\pi - \alpha\}. \end{aligned}$$

By Montel's theorem, $\mathcal{F}$ is normal in $\mathbb{D} \setminus (L_1 \cup L_0 \cup L_{-1})$. Thus we only have to prove that $\mathcal{F}$ is normal on $\mathbb{D} \cap L_j \setminus \{0\}$ for $j \in \{0, \pm 1\}$.

First we prove that $\mathcal{F}$ is normal on $\mathbb{D} \cap L_0 \setminus \{0\}$. In order to do so, suppose that $\mathcal{F}$ is not normal at some point $z_0 \in L_0 \setminus \{0\}$. Applying Theorem D to the family $\{1 - f \colon f \in \mathcal{F}\}$ we see that there exists a sequence (f_k) in $\mathcal{F}$ such that either $f_k|_{S^+} \to 1$ and $f_k|_{S^-} \to \infty$ or $f_k|_{S^+} \to \infty$ and $f_k|_{S^-} \to 1$. Without loss of generality we may assume that the first alternative holds. If (f_k) is not normal at some $z_1 \in L_1 \setminus \{0\}$, then – again by Theorem D –

there exists a subsequence of (f_k) which tends to 0 or to ∞ in S^+. This is incompatible with our previous assumption that $f_k|_{S^+} \to 1$. Hence (f_k) is normal on $\mathbb{D}\cap L_1\setminus\{0\}$. We conclude that (f_k) is normal in $S^+\cup S\cup L_1\setminus\{0\}$ and hence that $f_k|_{S^+\cup S\cup L_1\setminus\{0\}} \to 1$. In particular, $f_k|_S \to 1$. On the other hand, $f_k|_{S^-} \to \infty$. Hence (f_k) is not normal at any point of L_{-1}. Since $f_k|_{S^-} \to \infty$ we can now deduce from Theorem D that $f_k|_S \to 0$. This contradicts our previous finding that $f_k|_S \to 1$. Thus $\mathcal{F}$ is normal on $L_0 \setminus \{0\}$. Putting $T = S^+ \cup S^- \cup L_0 \setminus \{0\}$ we conclude that $\mathcal{F}$ is normal in T.

Suppose now that $\mathcal{F}$ is not normal at some point $z_0 \in \mathbb{D} \setminus \{0\}$. It follows that $z_0 \in (L_1 \cup L_{-1}) \setminus \{0\}$. Without loss of generality we may assume that $z_0 \in L_1 \setminus \{0\}$. Theorem D implies that there exists a sequence (f_k) in $\mathcal{F}$ such that either $f_k|_S \to \infty$ and $f_k|_T \to 0$ or $f_k|_S \to 0$ and $f_k|_T \to \infty$. In particular we see that the sequence (f_k) is not normal at any point of $L_1 \cup L_{-1}$. We begin by considering the case that the first of the two above possibilities holds; that is, $f_k|_S \to \infty$ and $f_k|_T \to 0$.

We define $u_k\colon \mathbb{D} \to [-\infty, \infty)$,

$$u_k(z) = \frac{\log|f_k(z)|}{\log|f_k(\frac{1}{2})|}.$$

We will prove that the sequence (u_k) is locally bounded in $\mathbb{D}$. Once this is known, we can deduce (see, for example, [9, Theorems 4.1.8, 4.1.9] or [10, Theorems 3.2.12, 3.2.13]) that some subsequence of (u_k) converges to a limit function u which is subharmonic in $\mathbb{D}$ and harmonic in $\mathbb{D} \setminus L_0$. Moreover, $u(z) > 0$ for $z \in S$ while $u(z) \le 0$ for $z \in \mathbb{D} \setminus S$.

Lemma 2.5 now implies that $\alpha \ge \pi/2$ and that $\beta = \pi$ if $\alpha > \pi/2$. The conclusion then follows since if $\alpha = \pi/2$, then $\angle(L_{-1}, L_1) = 2\alpha = \pi$, while if $\alpha > \pi/2$ and thus $\beta = \pi$, then $\angle(L_0, L_1) = \beta - \alpha = \pi - \alpha < \pi/2$ and $\angle(L_0, L_{-1}) = 2\pi - \alpha - \beta = \pi - \alpha = \angle(L_0, L_1)$.

In order to prove that (u_k) is locally bounded, let $0 < \varepsilon < 1/8$. Proposition 1.1 yields that, for sufficiently large k, there exist simple closed curves Γ_k in $\{z\colon 1-\varepsilon/2 < |z| < 1-\varepsilon/4\}$ and γ_k in $\{z\colon \varepsilon/2 < |z| < \varepsilon\}$ such that $|f_k(z)| > 1$ for $z \in (\Gamma_k \cup \gamma_k) \cap S$ while $|f_k(z)| < 1$ for $z \in (\Gamma_k \cup \gamma_k) \cap T$. Moreover, both Γ_k and γ_k surround 0 and they intersect L_1 and L_{-1} only once, at 1-points of f_k. In fact, these curves can be constructed by taking small segments orthogonal to L_1 and L_{-1}, and connecting the endpoints of these segments within the intersection of S and T with the corresponding annuli.

Let D_k be the domain between γ_k and Γ_k and let X_k be the set of all

$z \in \overline{D_k}$ for which $|f_k(z)| = 1$. Then both $X_k \cap \Gamma_k$ and $X_k \cap \gamma_k$ consist of two 1-points of f_k. Let U_k be the component of $D_k \setminus X_k$ which contains $\frac{1}{2}$. Next, for large k we have $|f_k(z)| < 1$ for $z \in L_0$ with $\varepsilon/2 \le |z| \le 1-\varepsilon/4$ and hence in particular for $z \in L_0 \cap D_k$. Let V_k be the component of $D_k \setminus X_k$ which contains $L_0 \cap D_k$. Then, for large k, we have $|f_k(z)| > 1$ for $z \in U_k$ while $|f_k(z)| < 1$ for $z \in V_k$.

We claim that $D_k \setminus X_k = U_k \cup V_k$. Indeed, let W be a component of $D_k \setminus X_k$ which is different from U_k and V_k. Since $(\Gamma_k \cup \gamma_k) \cap S \subset \partial U_k$ and $(\Gamma_k \cup \gamma_k) \cap T \subset \partial V_k$ we have $\partial W \subset X_k$ for large k. By the maximum principle, we thus have $|f_k(z)| < 1$ for $z \in W$. The minimum principle now yields that W contains a zero of f_k. Hence W and thus ∂W intersect $L_0 \cap \overline{D_k}$, which is a contradiction for large k, since $\partial W \subset X_k$ and thus $|f_k(z)| = 1$ for $z \in \partial W$, but $f_k|_{L_0 \cap \overline{D_k}} \to 0$. Thus $D_k \setminus X_k = U_k \cup V_k$ as claimed. We also conclude that X_k consists of two analytic curves $\sigma_{1,k}$ and $\sigma_{-1,k}$, which are close to the rays L_1 and L_{-1}.

We now prove that (u_k) is bounded in $\overline{D}(0, 1-\varepsilon)$. In order to do so we choose $c_k \in \partial D(0, 1-\varepsilon)$ such that

$$u_k(c_k) = \max_{|z|=1-\varepsilon} u_k(z).$$

Clearly, $c_k \in U_k$ for large k. For $j \in \{1,2,3,4\}$, we put $r_j = 1 - \varepsilon j/4$. Thus $|c_k| = 1-\varepsilon = r_4$. Similar to the curve Γ_k in $\{z \colon r_2 < |z| < r_1\}$ there exists a closed curve Γ'_k in $\{z \colon r_4 < |z| < r_3\}$ which surrounds 0 such that $|f_k(z)| > 1$ for $z \in \Gamma'_k \cap S$ while $|f_k(z)| < 1$ for $z \in \Gamma'_k \cap T$. Thus $\Gamma'_k \cap S \subset U_k$ and $\Gamma'_k \cap T \subset V_k$.

By the maximum principle, there exists a curve ξ_k in U_k which connects c_k with $\partial\mathbb{D}$ and on which u_k is bigger than $u_k(c_k)$. Let τ_k be a part of ξ_k which connects $\partial D(0, r_3)$ with $\partial D(0, r_2)$ and, except for its endpoints, is contained in $\{z \colon r_3 < |z| < r_2\}$; see Figure 1. Then $|u_k(z)| \ge u_k(c_k)$ for $z \in \tau_k$. Let $e_{k,j}$ be the endpoint of τ_k on $\partial D(0, r_j)$, for $j \in \{2,3\}$. Without loss of generality we may assume that the distance of $e_{k,3}$ to L_{-1} is less than or equal to the distance to L_1, which means that $\operatorname{Im} e_{k,3} \le 0$.

We define a domain G_k as follows; cf. Figure 1. If τ_k does not intersect the segment $\{re^{i(\alpha-\varepsilon)} \colon r_3 \le r \le r_2\}$, let G_k be the domain bounded by the segments $\{re^{-i(\alpha-\varepsilon)} \colon \frac{1}{4} \le r \le r_3\}$ and $\{re^{i(\alpha-\varepsilon)} \colon \frac{1}{4} \le r \le r_2\}$, the arc $\{\frac{1}{4}e^{i\theta} \colon |\theta| \le \alpha - \varepsilon\}$, the arc of $\partial D(0, r_3)$ that connects $e_{k,3}$ and $r_3 e^{-i(\alpha-\varepsilon)}$ in $\{r_3 e^{i\theta} \colon |\theta| \le \alpha + \varepsilon\}$, the arc of $\partial D(0, r_2)$ that connects $e_{k,2}$ and $r_2 e^{i(\alpha-\varepsilon)}$ in $\{r_2 e^{i\theta} \colon |\theta| \le \alpha + \varepsilon\}$, and the curve τ_k.

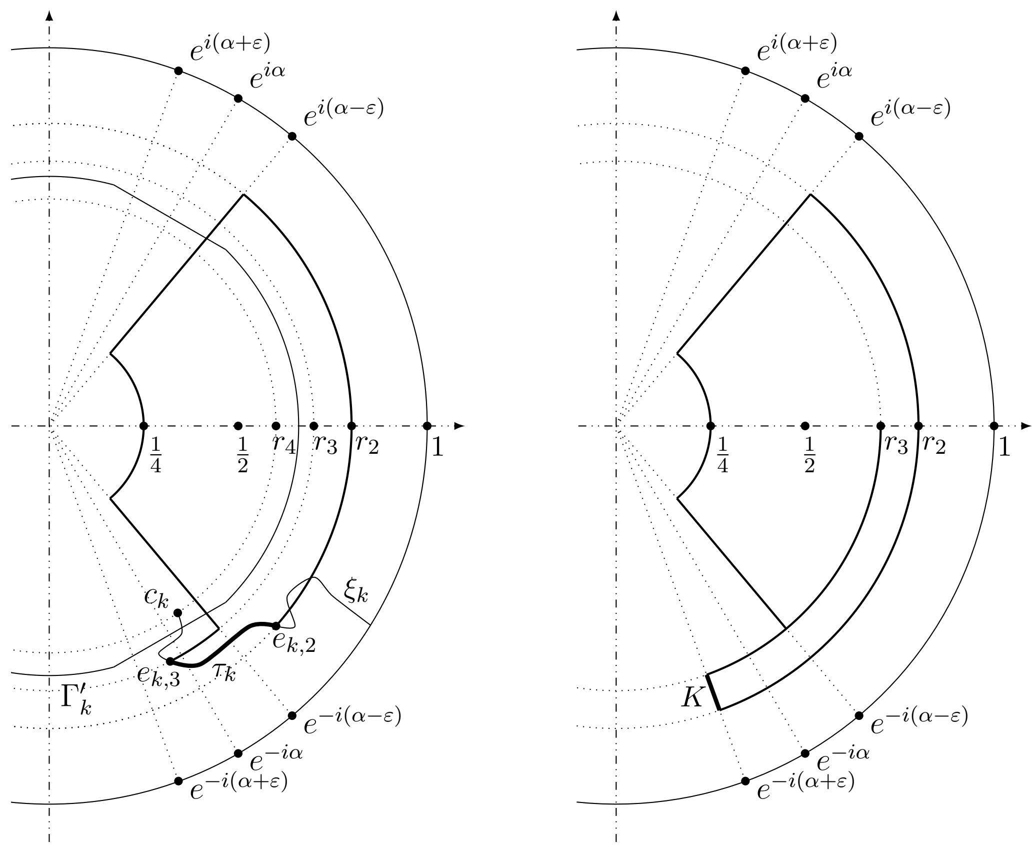

Figure 1: The curves ξ_k, τ_k and Γ'_k and the domains G_k (left) and H (right).

If τ_k intersects the segment $\{re^{i(\alpha-\varepsilon)}: r_3 \leq r \leq r_2\}$, let d_k denote the first point of intersection so that the part τ'_k of τ_k which is between $e_{k,3}$ and d_k is contained in $\{re^{i\theta}: r_3 < r < r_2, -\alpha - \varepsilon < \theta < \alpha - \varepsilon\}$. We then define G_k as the domain bounded by the the curve τ'_k, the segment $\{re^{i(\alpha-\varepsilon)}: \frac{1}{4} \leq r \leq |d_k|\}$ and – as before – the arc $\{\frac{1}{4}e^{i\theta}: |\theta| \leq \alpha - \varepsilon\}$, the segment $\{re^{-i(\alpha-\varepsilon)}: \frac{1}{4} \leq r \leq r_3\}$ and the arc of $\partial D(0, r_3)$ that connects $e_{k,3}$ and $r_3 e^{-i(\alpha-\varepsilon)}$ in $\{r_3 e^{i\theta}: |\theta| \leq \alpha + \varepsilon\}$.

We claim that $G_k \subset U_k$ for large k. In order to prove this it suffices to prove that $\partial G_k \subset U_k$. We restrict to the case that τ_k does not intersect the segment $\{re^{i(\alpha-\varepsilon)}: r_3 \leq r \leq r_2\}$, since the other case is similar. First we note that the segments $\{re^{-i(\alpha-\varepsilon)}: \frac{1}{4} \leq r \leq r_3\}$ and $\{re^{i(\alpha-\varepsilon)}: \frac{1}{4} \leq r \leq r_2\}$ as well as the arc $\{\frac{1}{4}e^{i\theta}: |\theta| \leq \alpha - \varepsilon\}$ are clearly in U_k for large k, since $f_k|_S \to \infty$ as $k \to \infty$. Since ξ_k is in U_k and τ_k is a subcurve of ξ_k, the curve τ_k is also in U_k.

It remains to show that the arc of $\partial D(0, r_3)$ that connects $e_{k,3}$ and $r_3 e^{-i(\alpha-\varepsilon)}$ is in U_k. If this is not the case, then this arc must intersect ∂U_k and thus must intersect the curve $\sigma_{-1,k}$, which constitutes the part of ∂U_k that is near L_{-1}. Since ξ_k is in U_k this means that $\sigma_{-1,k}$ must also intersect Γ'_k, at a point between the intersections of Γ'_k with ξ_k and with the positive real axis. But this part of Γ'_k is in U_k, since $\Gamma'_k \cap S \subset U_k$ and $\Gamma'_k \cap T \subset V_k$. Hence $\sigma_{-1,k}$ does not intersect the arc connecting $e_{k,3}$ and $r_3 e^{-i(\alpha-\varepsilon)}$ and thus this arc is in U_k. Similarly, we see that the arc of $\partial D(0, r_2)$ that connects $e_{k,2}$ and $r_2 e^{i(\alpha-\varepsilon)}$ is in U_k. Altogether thus $G_k \subset U_k$ for large k.

Let

$$\begin{aligned} H = &\{re^{i\theta} \colon \tfrac{1}{4} < r < r_3, |\theta| < \alpha - \varepsilon\} \cup \{r_3 e^{i\theta} \colon 0 < \theta < \alpha - \varepsilon\} \\ &\cup \{re^{i\theta} \colon r_3 < r < r_2, -\alpha - \varepsilon < \theta < \alpha - \varepsilon\} \end{aligned}$$

and let $K = \{re^{i(-\alpha-\varepsilon)} \colon r_3 \le r \le r_2\} \subset \partial H$; see Figure 1. Then $G_k \subset H$.

It follows from Lemma 2.6 and the configuration of the domains G_k and H that $\omega(z, K, H) \le \omega(z, \tau_k, G_k)$ for $z \in G_k$. In particular, $\omega(\frac{1}{2}, K, H) \le \omega(\frac{1}{2}, \tau_k, G_k)$. On the other hand, since $G_k \subset U_k$ and thus $u_k(z) \ge 0$ for $z \in \partial G_k$ while $u_k(z) \ge u_k(c_k)$ for $z \in \tau_k$ it follows that $u_k(z) \ge u_k(c_k)\omega(z, \tau_k, G_k)$ for $z \in G_k$. Altogether we thus have

$$1 = u_k(\tfrac{1}{2}) \ge u_k(c_k)\omega(\tfrac{1}{2}, \tau_k, G_k) \ge u_k(c_k)\omega(\tfrac{1}{2}, K, H).$$

It follows that

$$\max_{|z|=1-\varepsilon} u_k(z) = u_k(c_k) \le \frac{1}{\omega(\frac{1}{2}, K, H)}$$

so that (u_k) is bounded in $\overline{D}(0, 1-\varepsilon)$. Since $\varepsilon > 0$ can be taken arbitrarily small, we conclude that (u_k) is locally bounded in $\mathbb{D}$. This completes the proof in the case that $f_k|_S \to \infty$ and $f_k|_T \to 0$.

It remains to consider the case that $f_k|_S \to 0$ and $f_k|_T \to \infty$. Since $L_0 \setminus \{0\} \subset T$ we conclude that if $\varepsilon > 0$, then, for large k, the function f_k has no zeros in $\{z \colon \varepsilon < |z| < 1 - \varepsilon\}$. Thus u_k is harmonic there. As before we see that the sequence (u_k) is locally bounded so that some subsequence of it converges to a limit u which is subharmonic in $\mathbb{D}$. But now u is actually harmonic in $\mathbb{D} \setminus \{0\}$. The conclusion follows again from Lemma 2.5 which yields that $\alpha = \pi/2$ and hence $\angle(L_{-1}, L_1) = 2\alpha = \pi$. $\square$

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