MEROMORPHIC FUNCTIONS WITH TWO COMPLETELY INVARIANT DOMAINS

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Dedicated to the memory of Professor I. N. Baker

ABSTRACT. We show that if a meromorphic function has two completely invariant Fatou components and only finitely many critical and asymptotic values, then its Julia set is a Jordan curve. However, even if both domains are attracting basins, the Julia set need not be a quasicircle. We also show that all critical and asymptotic values are contained in the two completely invariant components. This need not be the case for functions with infinitely many critical and asymptotic values.

1. INTRODUCTION AND MAIN RESULT

Let f be a meromorphic function in the complex plane $\mathbb{C}$. We always assume that f is not fractional linear or constant. For the definitions and main facts of the theory of iteration of meromorphic functions we refer to a series of papers by Baker, Kotus and Lü [2, 3, 4, 5], who started the subject, and to the survey article [8]. For the dynamics of rational functions we refer to the books [7, 11, 20, 24].

A component D of the set of normality is called *completely invariant* if $f^{-1}(D) = D$. There is an unproved conjecture (see [4, p. 608], [8, Question 6]) that a meromorphic function can have at most two completely invariant domains. For rational functions this fact easily follows from Fatou's investigations [14], and it was first explicitly stated by Brolin [10, §8]. Moreover, if a rational function has two completely invariant domains, then their common boundary is a Jordan curve on the Riemann sphere, and each of the domains coincides with with the basin of attraction of an attracting or superattracting fixed point, or of an attracting petal of a neutral fixed point with multiplier 1; see [14, p. 300-303] and [10]. All critical values of f are contained in the completely invariant domains.

In this paper we extend these results to a class of transcendental meromorphic functions in $\mathbb{C}$. This class S consists of meromorphic functions with finitely many critical and asymptotic values. Let $A = A(f)$ be the set of critical and asymptotic values. We also call the elements of A *singular values* of f. For $f \in S$ the map

$$f : \mathbb{C}\backslash f^{-1}(A) \to \overline{\mathbb{C}}\backslash A$$

is a covering. By $J = J(f) \subset \mathbb{C}$ we denote the Julia set of f.

Baker, Kotus and Lü [4, Theorem 4.5] proved that functions of the class S have at most two completely invariant domains. We complement their result with the following

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Theorem. *Let f be a function of the class S, having two completely invariant domains D_j, $j = 1, 2$. Then*
(i) each D_j is the basin of attraction of an attracting or superattracting fixed point, or of a petal of a neutral fixed point with multiplier 1,
(ii) $A(f) \subset D_1 \cup D_2$,
(iii) each D_j contains at most one asymptotic value, and if a is an asymptotic value and $0 < \epsilon < \operatorname{dist}(a, A \backslash \{a\})$, then the set $\{z : |f(z) - a| < \epsilon\}$ has only one unbounded component,
(iv) $J \cup \{\infty\}$ is a Jordan curve in $\overline{\mathbb{C}}$.

A simple example of a meromorphic function of class S with two completely invariant domains is $f(z) = \tan z$, for which the upper and lower half-planes are completely invariant, and each of these half-planes is attracted to one of the two petals of the fixed point $z = 0$.

More examples will be given later in §3.

In the case that f is rational and both D_1 and D_2 are attracting or superattracting basins, Sullivan [26, Theorem 7] and Yakobson [27] proved that J is a quasicircle. Steinmetz [25] extended this result to the case that both completely invariant domains are basins of two petals attached to the same neutral fixed point. We will construct examples of transcendental functions in S for which D_1 and D_2 are attracting basins, or basins of petals attached to the same neutral fixed point, but where J is not a quasicircle; see Examples 1 and 2 in §3.

On the other hand, Keen and Kotus [15, Corollary 8.2] have shown that for the family $f_\lambda(z) = \lambda \tan z$ there exists a domain Ω containing $(1, \infty)$ such that f_λ has two completely invariant attracting basins and $J(f_\lambda)$ is a quasicircle for $\lambda \in \Omega$. Meromorphic functions for which the Julia set is contained in a quasicircle were also considered by Baker, Kotus and Lü [2, §5].

Baker [1] proved that an entire function f can have at most one completely invariant component of the set of normality, and that such a domain contains all critical values. Eremenko and Lyubich [13, §6] proved that a completely invariant domain of an entire function also contains all asymptotic values of a certain type, namely those associated with direct singularities of f^{-1}. On the other hand, Bergweiler [9] constructed an entire function with a completely invariant domain D, and such that some asymptotic value belongs to the Julia set $J = \partial D$. We modify this example to construct a meromorphic function with two completely invariant components of the set of normality, which has an asymptotic value on the Julia set; see Examples 3 and 4 in §3. So (ii) does not hold for general meromorphic functions, without the assumption that $f \in S$.

2. Proof of the Theorem

We shall need the following result of Baker, Kotus and Lü [4, Lemmas 4.2 and 4.3] which does not require that $f \in S$. Here and in the following all topological notions are related to $\mathbb{C}$ unless $\overline{\mathbb{C}}$ is explicitly mentioned.

Lemma 1. *Let f be meromorphic with two completely invariant components D_1 and D_2 of the set of normality. Then D_1 and D_2 are simply-connected and $J = \partial D_1 = \partial D_2$. In particular, J is a connected subset of $\mathbb{C}$.*

Proof of the Theorem. As the result is known for rational f, we assume that our f is transcendental.

Statement (i) follows from the classification of dynamics on the Fatou set for meromorphic functions of the class S, given in [5, Theorems 2.2 and 2.3], [8, Theorem 6], and [23, p. 3252].

To prove (ii), we consider the finite sets $A_i = A \cap D_i$. For $j = 1, 2$, let Γ_j be a Jordan curve in D_j, which separates A_j from ∂D_j. Let G_j be the Jordan regions bounded by the Γ_j. Let $G = \overline{\mathbb{C}} \backslash (\overline{G_1} \cup \overline{G_2})$ be the doubly connected region in $\overline{\mathbb{C}}$ bounded by Γ_1 and Γ_2. Notice that G contains the Julia set J.

We denote $\gamma_j = f^{-1}(\Gamma_j)$. Then

$$f : \gamma_j \to \Gamma_j, \; j = 1, 2, \tag{1}$$

are covering maps.

We claim that each $\gamma_j \subset \mathbb{C}$ is a single simple curve tending to infinity in both directions, which means that $\gamma_j \cup \{\infty\}$ is a Jordan curve in $\overline{\mathbb{C}}$.

To prove the claim, we fix j and consider the full preimages $H_j = f^{-1}(G_j)$ and $F_j = f^{-1}(D_j \backslash \overline{G_j})$. Then $D_j = F_j \cup H_j \cup \gamma_j$. The boundary of each component of F_j contains a component of γ_j, and this gives a bijective correspondence between components of F_j and components of γ_j.

We notice that H_j is connected. Indeed, by complete invariance of D_j, we have $H_j \subset D_j$, so every two points z_1 and z_2 in H_j can be connected by a curve β in D_j, so that β does not pass through the critical points of f. The image of this curve, $f(\beta)$ begins and ends in G_j, and does not pass through the critical values of f. By a small perturbation of β we achieve that $f(\beta)$ does not pass through asymptotic values. Using the fact that

$$f : D_j \backslash f^{-1}(A_j) \to D_j \backslash A_j,$$

is a covering and that $A_j \subset G_j$, we can deform β into a curve in H_j which still connects z_1 and z_2. This proves that the H_j are connected.

It follows that H_j is unbounded, as it contains infinitely many preimages of a generic point in G_j.

It is easy to see that the boundary of each component F_j' of F_j intersects the Julia set.

For each component F_j' of F_j, the intersection $\partial F_j' \cap \overline{H_j}$ is a component γ_j' of γ_j. This component γ_j' divides the plane into two parts, one containing H_j and another containing F_j'. We conclude that every component of γ_j is unbounded, because H_j is unbounded, and ∂F_j intersects the Julia set which is unbounded and connected by Lemma 1. (A similar argument for unboundedness of each component of γ_j was given in [4]).

For every component γ_j' of γ_j, the component of $\mathbb{C} \backslash \gamma_j'$ that contains F_j' intersects the Julia set. Since the Julia set is connected by Lemma 1, we conclude that F_j and γ_j are connected. So the map (1) is a universal covering by a connected set γ_j, for each $j = 1, 2$. Thus $\gamma_1 \cup \{\infty\}$ and $\gamma_2 \cup \{\infty\}$ are Jordan curves in $\overline{\mathbb{C}}$ whose intersection consists of the single point ∞. This proves our claim.

As a corollary we obtain that the point ∞ is accessible from each D_j, and so all poles of all iterates f^n are accessible from each D_j. (This fact was established in [4].)

Next we note that the set $\gamma_1 \cup \gamma_2 \cup \{\infty\}$ separates the sphere into three simply connected regions. We denote by W that region whose boundary in $\overline{\mathbb{C}}$ is $\gamma_1 \cup \gamma_2 \cup$

$\{\infty\}$. Then

$$f^{-1}(G) = W, \tag{2}$$

in particular, W contains the Julia set J.

To prove (ii) we choose an arbitrary point $w \in J$ and show that w is neither a critical value nor an asymptotic value.

Fix an arbitrary point $w_1 \in \Gamma_1$. The preimage $f^{-1}(w_1)$ consists of infinitely many points $a_k \in \gamma_1$, which we enumerate by all integers in a natural order on γ_1. Let ϕ_k be the branches of f^{-1} such that $\phi_k(w_1) = a_k$. We find a simple curve Δ from w_1 to some point $w_2 \in \Gamma_2$ such that $\Delta \backslash \{w_1, w_2\}$ is contained in $G \backslash \{w\}$, and such that all branches ϕ_k have analytic continuation along Δ to the point w_2. We denote

$$G' = G \backslash \Delta \subset \overline{\mathbb{C}}.$$

The full preimage $f^{-1}(\Delta)$ consists of infinitely many disjoint simple curves δ_k starting at the points a_k and ending at some points $b_k \in \gamma_2$. The open curves $\delta_k \backslash \{a_k, b_k\}$ are disjoint from $\gamma_1 \cup \gamma_2$.

For every integer k, let Q_k be the Jordan region bounded by δ_k, δ_{k+1}, the arc (a_k, a_{k+1}) of γ_1 and the arc (b_k, b_{k+1}) of γ_2. Then f maps Q_k into G', and $f(\partial Q_k) \subset \partial G'$. So

$$f : Q_k \to G' \tag{3}$$

is a ramified covering, continuous up to the boundary. Furthermore, the boundary map is a local homeomorphism. As each point of $\Gamma_1 \backslash \{w_1\}$ has only one preimage on ∂Q_k, we conclude that (3) is a homeomorphism. Now it follows that the restriction $f : (b_k, b_{k+1}) \to \Gamma_2 \backslash \{w_2\}$ is a homeomorphism, in particular,

$$W = \bigcup_{k=-\infty}^{\infty} Q_k.$$

It follows that there are no critical points over w, so w is not a critical value.

If w were an asymptotic value, there would be a curve α in W which tends to infinity, and such that $f(z) \to w$ as $z \to \infty$, $z \in \alpha$. But this curve α would intersect infinitely many of the curves δ_k, so its image $f(\alpha)$ would intersect Δ infinitely many times, which contradicts the assumption that $f(\alpha)$ tends to w.

This completes the proof of (ii). The proof actually shows that $f : W \to G$ is a universal covering, a fact which we will use later.

To prove (iii), let us assume that D_1 contains two asymptotic values, or that $\{z \in D_1 : |f(z) - a| < \epsilon\}$ has two unbounded components for some asymptotic value $a \in D_1$. Then there exists a curve $\alpha \subset D_1$, tending to infinity in both directions, such that $f(z)$ has limits as $z \to \infty$, $z \in \alpha$, in both directions, where these limits are the two asymptotic values in the first case, and where both limits are equal to a in the second case, but the two tails of the curve α are in different components of $\{z \in D_1 : |f(z) - a| < \epsilon\}$. Now one of the regions, say R, into which α partitions the plane does not intersect the Julia set J (because J is connected by Lemma 1), and thus $R \subset D_1$. We want to conclude that f has a limit as $z \to \infty$ in R.

To do this, we choose an arbitrary point $b \in D_2$ and consider the function $g(z) = (f - b)^{-1}$ which is holomorphic and bounded in D_1. This function has limits when $z \to \infty$, $z \in \alpha$, so by a theorem of Lindelöf [21], these limits coincide and g has a limit as $z \to \infty$ in R. This proves (iii).

To prove (iv), we distinguish several cases, according to the dynamics of f in each D_j.

1. Suppose first that both D_1 and D_2 are basins of attraction of attracting or superattracting points. Then we choose the curves Γ_j as above, but with the additional property that $f(\Gamma_j) \subset G_j$, that is $f(\Gamma_j) \cap \overline{G} = \emptyset$. To achieve this, we denote by z_j the attracting or superattracting fixed point in D_j, choose G_j to be the open hyperbolic disc centered at z_j, of large enough hyperbolic radius, so that $A_j \subset G_j$, and put $\Gamma_j = \partial G_j$. Then the G_j are $f-$invariant, and moreover $f(\overline{G_j}) \subset G_j$ for $j = 1, 2$, because f is strictly contracting the hyperbolic metric in D_j. It follows that the closure of $W = f^{-1}(G)$ is contained in G. Let h be the hyperbolic metric in G, and $|f'(z)|_h$ the infinitesimal length distortion by f at the point $z \in W$ with respect to h. By the Theorem of Pick [20, Theorem 2.11] there exists $K > 1$ such that

$$|f'(z)|_h \geq K, \quad z \in W. \tag{4}$$

Now we consider successive preimages $W_n = f^{-n}(W)$. Every component of W_n is a Jordan domain whose boundary consists of two cross-cuts, one of D_1 another of D_2. These crosscuts meet at two poles of f^n. It follows from (4) that the diameter (with respect to the metric h) of every component of W_n is at most CK^{-n}, where $C > 0$ is a constant. Now we notice that

$$J = \bigcap_{n=1}^{\infty} \overline{W_n}$$

and prove that every point $z \in J$ is accessible both from D_1 and D_2.

The accessibility of poles of the iterates f^n was already noticed before. Now we assume that z is not a pole of any iterate. Let V_n be the component of W_n that contains z. Then $V_1 \supset V_2 \supset \ldots$. The intersection $V_k \cap D_j$ is connected (its relative boundary with respect to D_j is a cross-cut in D_j), so one can choose a sequence $z_{k,j} \in V_k \cap D_j$ and connect $z_{k,j}$ with $z_{k+1,j}$ by a curve $\ell_{k,j}$ in $V_k \cap D_j$. The union of these curves gives a curve in D_j which tends to z.

The proof of (iii) in the attracting case is completed by an application of Shoenflies' theorem that if each point of a common boundary of two domains on the sphere is accessible from both domains then this common boundary is a Jordan curve [22].

2. To prove (iv) in the remaining cases, suppose, for example, that D_1 is the domain of attraction of a petal associated with a neutral fixed point a. We need several lemmas.

Lemma 2. *There exists a Jordan domain G_1 with the properties $\overline{G_1} \subset D_1 \cup \{a\}$, $f(\overline{G_1}) \subset G_1 \cup \{a\}$, $A_1 \subset G_1$, and G_1 is absorbing, that is for every compact $K \subset D_1$ there exists a natural integer n such that $f^n(K) \subset G_1$.*

Proof. It is well known (see, e. g. [20, §10]) that there exists a domain G_1 having all properties mentioned except possibly $A_1 \subset G_1$. Such a domain is called an *attracting petal.*

Let P be an attracting petal. Choose a point $z_0 \in D_1$ and let $r > 0$ be so large that the open hyperbolic disc $B(z_0, r)$ of radius r centered at z_0 contains A_1, and

$z_1 = f(z_0) \in B(z_0, r)$. Then put

$$G_1' = P \cup \left(\bigcup_{k=0}^{\infty} B(z_k, r) \right), \quad z_k = f^k(z_0).$$

Then G_1 is absorbing because the petal P is absorbing. Notice that for every neighborhood V of a, all but finitely many discs $B(z_k, r)$ are contained in V. This easily follows from the comparison of the Euclidean and hyperbolic metrics near a, or, alternatively, from the local description of dynamics near a neutral fixed point with multiplier 1. It is easy to see that $f(\overline{G_1'}) \subset G_1' \cup \{a\}$. Now we fill the holes in G_1': let X be the unbounded component of $\overline{G_1'}$ and $G_1 = \mathbb{C} \backslash \overline{X}$. It is easy to see that G_1 is a Jordan domain (its boundary is a union of arcs of circles which is locally finite, except at the point a, plus some boundary arcs of the petal). □

Now we fix the following notations till the end of the proof of the Theorem. If D_j is a basin of an attracting or superattracting fixed point, let G_j be the Jordan region constructed in the first part of the proof of (iv). If D_j is a basin of a petal, let G_j be the region from Lemma 1. We define $\Gamma_j = \partial G_j$. This is a Jordan curve in D_j or in $D_j \cup \{a\}$ which encloses all singular values in D_j.

Next we define $G = \overline{\mathbb{C}} \backslash (\overline{G_1} \cup \overline{G_2})$. This region is simply connected in the case that both D_1 and D_2 are basins of two petals associated with the *same* fixed point, and doubly connected in all other cases. If G is doubly connected, we make a simple cut δ disjoint from the set A of singular points, as in the proof of (ii), to obtain a simply connected region $G' = G \backslash \delta$. If G is simply connected we set $G' = G$. All branches of f^{-n} are holomorphic in G'. Let $\gamma_j = f^{-1}(\Gamma_j)$.

Lemma 3. *There exists a repelling fixed point $b \in J$ which is accessible from both D_1 and D_2 by simple curves β_j which begin at some points of Γ_j and do not intersect G_j, and which satisfy $f(\beta_j) \cap \overline{G'} = \beta_j$, for $j = 1, 2$.*

Proof. We use the notation introduced before the statement of the Lemma. Fix one of the components Q, of $f^{-1}(G')$, such that $\overline{Q} \subset G'$. (These components are curvilinear quadrilaterals Q_k, as described in the proof of (ii)). Let ϕ be the branch of f^{-1} which maps G' onto Q. Then ϕ has an attracting fixed point $b \in Q$. Let $z_0 \in \Gamma_1$ $z_1 = \phi(z_0) \in \partial Q$. We connect z_0 and z_1 by a simple curve β in $(G' \backslash Q) \cap D_1$. Such curve exists because $z_1 \in \gamma_1$, and the component of $\mathbb{C} \backslash \gamma_1$ that contains G_1 is completely contained in D_1.

Now

$$\beta_1 = \bigcup_{k=1}^{\infty} \phi^k(\beta)$$

is a curve in D_1 tending to b which satisfies $f(\beta_1) \cap \overline{G'} = \beta_1$. Similarly a curve β_2 in D_2 is constructed. □

Now, if G is doubly connected, we set $G'' = G \backslash (\beta_1 \cup \beta_2 \cup \{b\})$. If G is simply connected then $G'' = G$. Then G'' is a simply connected region which contains no singular values of f. Let $\{\phi_k\}_{k \in \mathbb{N}}$ be the set of all branches of f^{-1} in G''. These branches map G'' onto Jordan regions $T_k \subset G''$. These regions T_k are of two types: the regions of the first type are contained in G'' with their closures, while the regions of the second type have common boundary points with G''.

We claim that there are only finitely many regions of the second type. To study these regions T_k, we first observe that the full preimage of Γ_j is a curve γ_j which can have at most one point in common with Γ_j, namely the neutral rational point on Γ_j. Thus the region $W = f^{-1}(G)$ bounded by γ_1 and γ_2 is a simply connected region contained in G, and the boundary ∂W has at most two common points with ∂G, namely the neutral rational points. The full preimage of the cross-cut

$$\alpha = \beta_1 \cup \beta_2 \cup \{b\}$$

constructed in Lemma 2 consists of countably many disjoint curves $\alpha_k \subset \overline{W}$. Each α_k connects a point on γ_1 to a point on γ_2. One of the α_k, say α_1, is contained in α while all others are disjoint from α. Thus our regions T_k are curvilinear rectangles, similar to the Q_k used in the proof of (ii). In particular, they cluster only at ∞ so that only finitely many of them are of the first type.

It is easy to see that every region of the second type has on its boundary exactly one of the following points: a neutral fixed point or the repelling fixed point b. Indeed, let T be a region of the second type, and $\phi : G'' \to T$ the corresponding branch of the inverse. Then the iterates $\phi^n(z)$ converge to a unique point $c \in \overline{T}$ by the Denjoy–Wolff Theorem. (This theorem is usually stated for the unit disk, but it follows for Jordan domains like T by the Riemann mapping theorem, using that the Riemann map extends homeomorphically to the boundary.) On the other hand, it follows from the local dynamics near the repelling fixed point b and a neutral fixed point a that there exists $\epsilon > 0$ such that $\phi^n(z) \to b$ if $|z - b| < \epsilon$ and $\phi^n(z) \to a$ if $|z - a| < \epsilon$, $z \in T$.

If ϕ_j and ϕ_k are two different branches of f^{-1} in G'', whose images are of the second type, then the images $(\phi_k \circ \phi_j)(G'')$ are compactly contained in G''. There exists a compact subset set $K \subset G''$ which contains all regions T of the first type as well as all images $(\phi_k \circ \phi_j)(G'')$ where $j \neq k$.

Now consider the hyperbolic metric in G'' and let $|\phi'|_h$ stand for the infinitesimal length distortion of a branch ϕ with respect to this hyperbolic metric. Then we have for all $z \in G''$ and some $\lambda \in (0, 1)$:

$$|\phi_k'(z)|_h < \lambda \quad \text{for all } k \text{ of the first type} \tag{5}$$

and

$$|(\phi_j \circ \phi_k)'(z)|_h < \lambda \quad \text{for all } j \neq k. \tag{6}$$

Let $W_n = f^{-n}(W)$. Then the Julia set can be represented as the intersection of a decreasing sequence of closed sets $J = \bigcap_{n=1}^{\infty} \overline{W_n}$. The points of the Julia set are divided into the following categories:

a) poles of f and their preimages,
b) neutral fixed points and their preimages,
c) the repelling point b and its preimages
d) those points of J which are interior to all $f^{-1}(G'')$.

We have already seen that all points of the categories a)-c) are accessible from each of the domains D_1 and D_2.

The proof that the points of the type d) are accessible is similar to the argument in the case that both D_1 and D_2 are attracting basins: we will show that each such point z can be surrounded by a nested sequence of Jordan curves whose diameter tends to zero.

Indeed, each point z of the class d) can be obtained as a limit

$$z = \lim_{n \to \infty} (\phi_{k_1} \circ \phi_{k_2} \circ \ldots \circ \phi_{k_n})(w),$$

where $w \in G''$. For a point z of the category d), the sequence $k_1, k_2, \ldots$ is uniquely defined. We will call this sequence the *itinerary* of z. Let us consider the domains

$$T_n(z) = (\phi_{k_1} \circ \phi_{k_2} \circ \ldots \circ \phi_{k_n})(G''),$$

in other words, $T_n(z)$ is that component of $f^{-n}(G'')$ which contains z. The boundary of $T_n(z)$ is a Jordan curve which intersects the Julia set at a finite set of points of categories a)-c). The complementary arcs of these points are cross-cuts of D_1 and D_2. Thus, to show that z is accessible from D_1 and D_2, it is enough to show that the diameter of $T_n(z)$ tends to zero as $n \to \infty$. Let $z \in J$ be a point of category d), and $k_1, k_2, \ldots$ its itinerary. Then the sequence $k_1, k_2, \ldots$ cannot have an infinite tail consisting of the branch numbers of the second type. Indeed, the iterates of any branch of the second type converge to a boundary point x of G'' (a neutral fixed point or the point b). In this case, z will be a preimage of x.

Now, assuming that the itinerary does not stabilize on a branch number of the second type, we use (5) and (6) to conclude that $\operatorname{diam} T_n(z) \to 0$.

This completes the proof. □

3. Examples

Example 1. Let

$$g(z) = \frac{1}{1 + a\cos\sqrt{z}}$$

where $0 < a < \frac{1}{5}$. Then there exists $b < 0$ such that

$$f(z) = \frac{g(z+b) - g(b)}{g'(b)}$$

has a parabolic fixed point at zero, with two completely invariant parabolic basins attached to it. Moreover, $f \in S$ and the Julia set of f is a Jordan curve, but not a quasicircle.

Verification. Note that g has no poles on the real axis. We have

$$g'(z) = \frac{a\sin\sqrt{z}}{\sqrt{z}(1 + a\cos\sqrt{z})^2}$$

and

$$g''(z) = -\frac{a^2(\cos\sqrt{z})^2 - a\cos\sqrt{z} - 2a^2}{z(1 + a\cos\sqrt{z})^3} - \frac{a\sin\sqrt{z}}{z\sqrt{z}(1 + a\cos\sqrt{z})^2}.$$

It follows that

$$\lim_{x\to-\infty} g''(x)x\cos\sqrt{x} = -\frac{1}{4a} < 0$$

so that $g''(x) > 0$ if x is negative and of sufficiently large modulus. On the other hand,

$$g''(0) = \frac{a(5a-1)}{2(a+1)^3} < 0.$$

Thus there exists $b \in (-\infty, 0)$ with $g''(b)=0$ and $g''(x) > 0$ for $x < b$.

The critical points of g are given by $(k\pi)^2$ where $k \in \mathbb{N}$, and g has a maximum there for odd k and a minimum for even k. It follows that $g'(x) > 0$ for $x < \pi^2$ and thus in particular for $x \le b$. Thus f has the critical points $(k\pi)^2 - b$, with corresponding critical values

$$d_\pm = \frac{(1 \pm a)^{-1} - g(b)}{g'(b)}.$$

Moreover, f has the asymptotic value $c = -g(b)/g'(b)$, which is also a Picard exceptional value of f, and no other asymptotic values. Thus $\operatorname{sing}(f^{-1}) = \{c, d_+, d_-\}$.

Next we note that $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$. Since $f'(x) = g'(x+b)/g'(b)$ we have $0 < f'(x) < 1$ for $x < 0$. It follows from the mean value theorem that if $x < 0$, then $x < f(x) < 0$. Thus $(-\infty, 0)$ lies in a parabolic basin U attached to the parabolic point b. In particular, U contains the value $c = -g(b)/g'(b)$ which is a Picard exceptional value of f. We note that $f^{-1}(D(0, r))$ is connected for sufficiently small $r > 0$, and thus U is completely invariant.

Since $f''(0) = 0$ there is at least one parabolic basin V different from U attached to the parabolic point 0. As f has a completely invariant domain, every component of the set of normality is simply connected. Thus V is simply connected. Now V must contain a singularity of f^{-1}. Thus V contains one of the critical values d_+ and d_-, and in fact a corresponding critical point $\xi = (k\pi)^2 - b$. Since $f^n(\xi) \in \mathbb{R} \cap V$ and $f^n(\xi) \to 0$ as $n \to \infty$, and since V is simply connected and symmetric with respect to the real axis, we conclude that $(0, \xi] \subset V$. Since $f((0, \infty)) \subset (0, d_-] = f(\pi^2 - b)$ we conclude that the positive real axis is contained in V.

We now show that V is completely invariant. Suppose that W is a component of $f^{-1}(V)$ with $W \neq V$. Since W contains no critical points of f, and V contains no asymptotic values, there exists a branch φ of f^{-1} which maps V to W. This functions φ can be continued analytically to any point in $\overline{\mathbb{C}} \setminus \{c\}$. By the monodromy theorem, φ extends to a a meromorphic function from $\overline{\mathbb{C}} \setminus \{c\}$ to $\mathbb{C}$. But this implies that f is univalent, a contradiction.

It follows from part (iv) of our Theorem 1 that the Julia set of f is a Jordan curve. On the other hand we note that if $w = u + iv$ with $|v| \leq T$, then $(\operatorname{Im}(w^2))^2 = (2uv)^2 \leq 4T^2u^2 \leq 4T^2(u^2 - v^2) + 4T^4 = 4T^2 \operatorname{Re}(w^2) + 4T^4$. It follows that if $4T^2 \operatorname{Re} z > (\operatorname{Im} z)^2 - 4T^4$, then $|\cos\sqrt{z}| \geq \sinh T$ and thus $z \in U$, if T is large enough. Thus the Julia set of f is contained in the domain $\{z \in \mathbb{C} : 4T^2 \operatorname{Re} z > (\operatorname{Im} z)^2 - 4T^4\}$. This implies that it is not a quasicircle.

Remark. It seems that g'' has only one negative zero. But since we have not proved this, we have just defined b to be the smallest zero of g''. The values a and b are related by

$$a = \frac{\sqrt{b}\cos\sqrt{b} - \sin\sqrt{b}}{\sqrt{b} + \sqrt{b}\sin^2\sqrt{b} - \sin\sqrt{b}\cos\sqrt{b}}$$

For example, if $b = -1$, then $a = 0.16763487\ldots$, $g(b) = 0.764166\ldots$ and $1/g'(b) = 16.083479\ldots$ so that

$$f(z) = 16.083479\left(\frac{1}{1 + 0.16763487\cos\sqrt{z-1}} - 0.764166\right)$$

Example 2. Let g and f be as in Example 1 and let $\alpha > 1$. Then there exists $\alpha_0 = \alpha_0(a) > 1$ such that if $1 < \alpha < \alpha_0$, then $f_\alpha(z) = \alpha f(z)$ has two completely invariant attracting basins.

Verification. It is not difficult to see that if α is sufficiently close to 1, then f_α does indeed have two attracting fixed points $\xi_+ > 0$ and and $\xi_- < 0$, with $\xi_\pm \to 0$ as $\alpha \to 1$. The verification that their immediate attracting basins are completely invariant is analogous to that in Example 1.

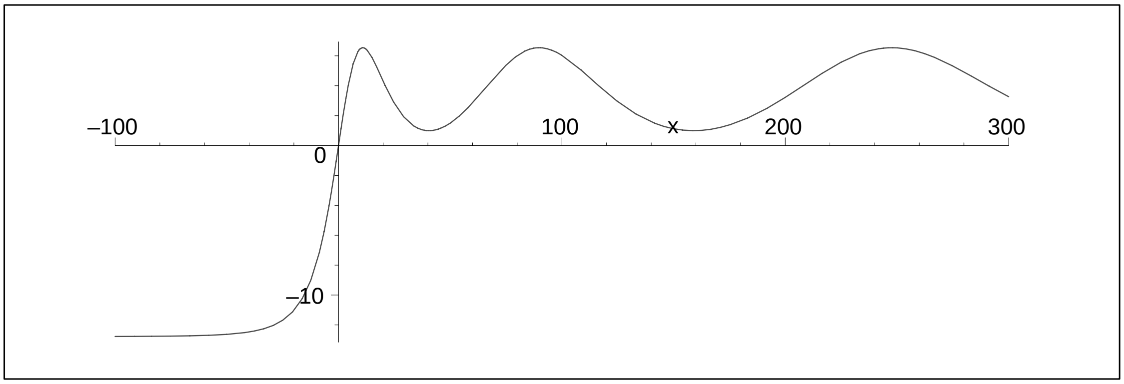

FIGURE 1. The graph of the function f from Example 1 with $b = -1$.

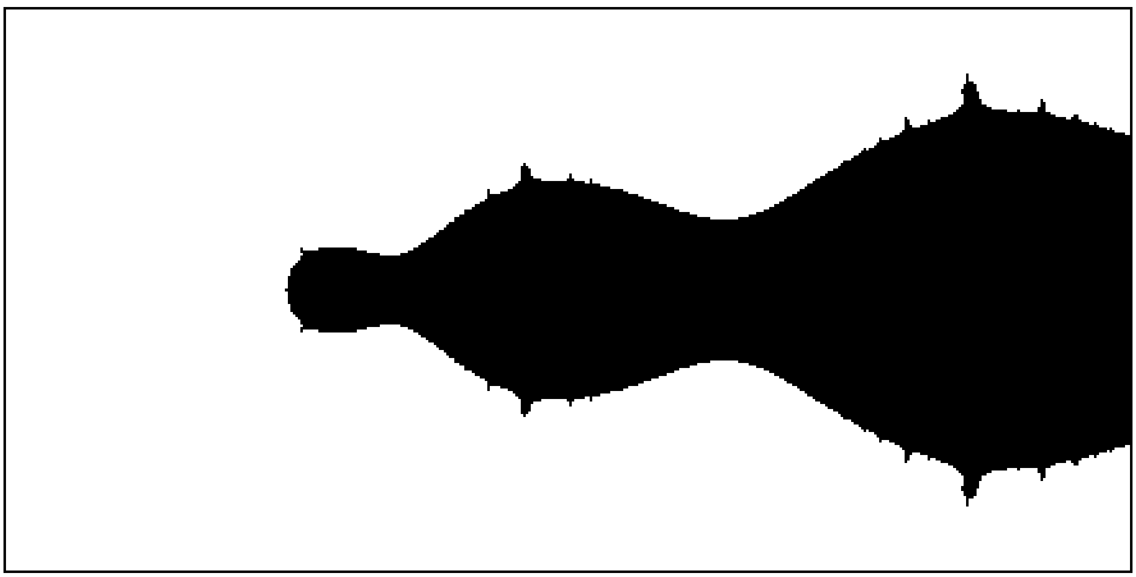

FIGURE 2. The immediate parabolic basin of the function f from Example 1 with $b = -1$ which contains the positive real axis is shown in black. The range shown is $-100 < \operatorname{Re} z < 300, |\operatorname{Im} z| < 100$.

Remark. We consider again the case $b = -1$. Then f_α has the form

$$f_\alpha(z) = \beta \left(\frac{1}{1 + 0.16763487 \cos \sqrt{z-1}} - 0.764166 \right)$$

with $\beta > 16.083479\ldots$. For $\beta = 26.712615\ldots$ the positive attracting fixed point coincides with the critical point $1 + \pi^2$ and thus is superattracting.

Example 3. Let

$$F(z) = \frac{12\pi^2}{5\pi^2 - 48} \left(\frac{(\pi^2 - 8)z + 2\pi^2}{z(4z - \pi^2)} \cos \sqrt{z} + \frac{2}{z} \right)$$

and

$$f(z) = \frac{F(z)}{1 + aF(z)}$$

where

$$a = -\frac{1}{15}\,\frac{7\,\pi^4 - 360\,\pi^2 + 2880}{(5\,\pi^2 - 48)\,\pi^2}.$$

Then 0 is a parabolic fixed point of f, with two completely invariant parabolic basins attached to it, one containing the positive real axis and one containing the negative real axis. Moreover, 0 is an asymptotic value of f contained in $J(f)$.

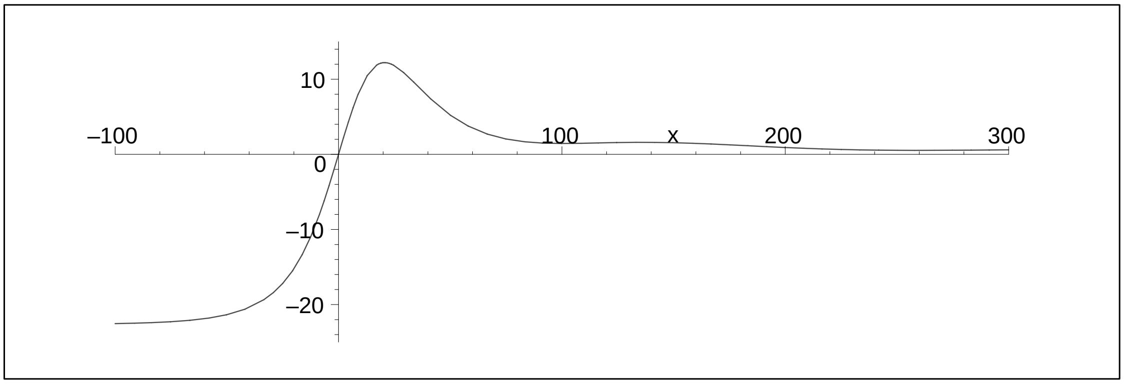

FIGURE 3. The graph of the function f from Example 3.

Verification. It was shown in [9] that F is entire and has a completely invariant parabolic basin attached to the parabolic point 0, and that 0 is an asymptotic value of F. The value a is chosen such that, besides $f(0) = 0$ and $f'(0) = 1$, we also have $f''(0) = 0$. Thus there are (at least) two parabolic basins attached to 0. Similarly as in Example 1 one can show, using arguments of [9] as well, that there is a parabolic basin containing the positive real axis and one containing the negative real axis, and that these are completely invariant. We omit the details.

Example 4. Let f be as in Example 1 and let $\alpha > 1$. Then there exists $\alpha_1 > 1$ such that if $1 < \alpha < \alpha_1$, then $f_\alpha(z) = \alpha f(z)$ has two attracting fixed points, one positive and one negative, the corresponding attracting basins are completely invariant, and 0 is an asymptotic value (and repelling fixed point) of f contained on the boundary of these attracting basins.

Example 5. For $a = -3.7488381 - 1.3843391i$ the function $f(z) = a \tan z / \tan a$ has fixed points $\pm a$ of multiplier 1. The Julia set is a Jordan curve by our theorem, but clearly not a quasicircle.

Example 6. For $a = 1/(1 - \tanh^2 1) = 2.3810978$ the function

$$f(z) = a \tan z - a \tan i + i$$

has the fixed point i of multiplier 1 and the attracting fixed point $-3.1864112i$. Again the Julia set is a Jordan curve, but not a quasicircle.

Example 7. Our final example has two completely invariant half-planes, but unlike $\tan z$, it has no asymptotic values. Another feature of this example is that it has minimal possible growth among the functions of class S, namely

$$T(r, f) = O((\log r)^2), \quad r \to \infty, \tag{7}$$

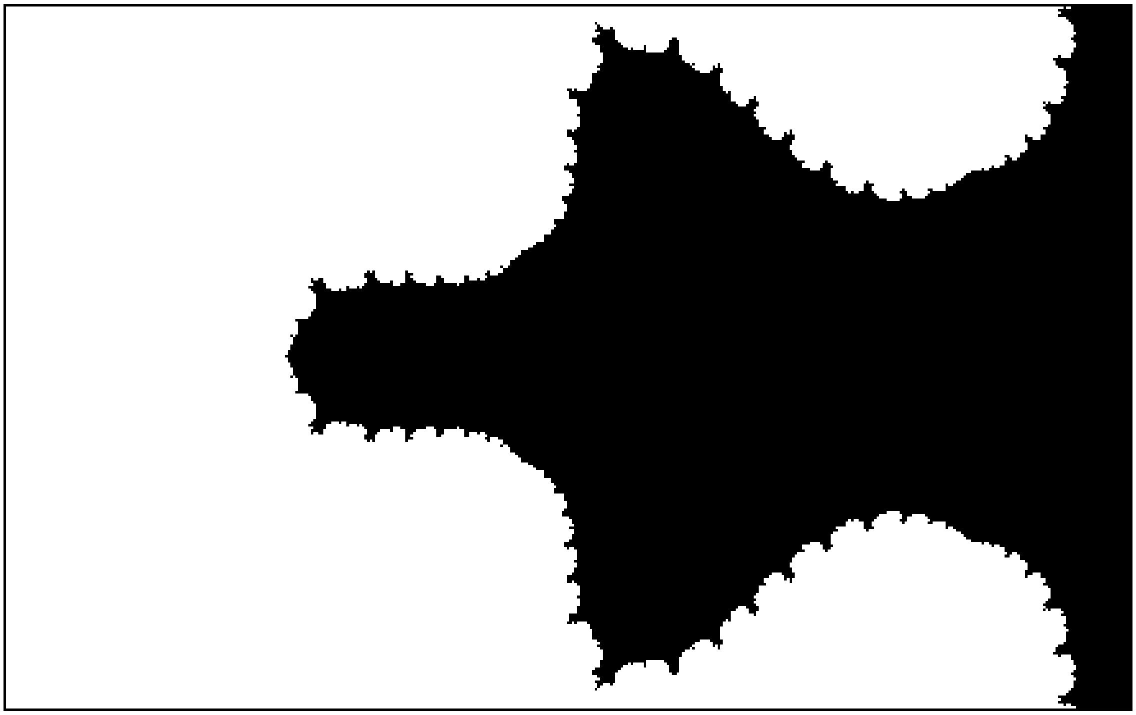

FIGURE 4. For $\alpha = 1.686035\ldots$ the positive attracting fixed point of the function from Example 4 is superattracting. Its attracting basin is shown in black. The range shown is $-100 < \operatorname{Re} z < 300, |\operatorname{Im} z| < 120$.

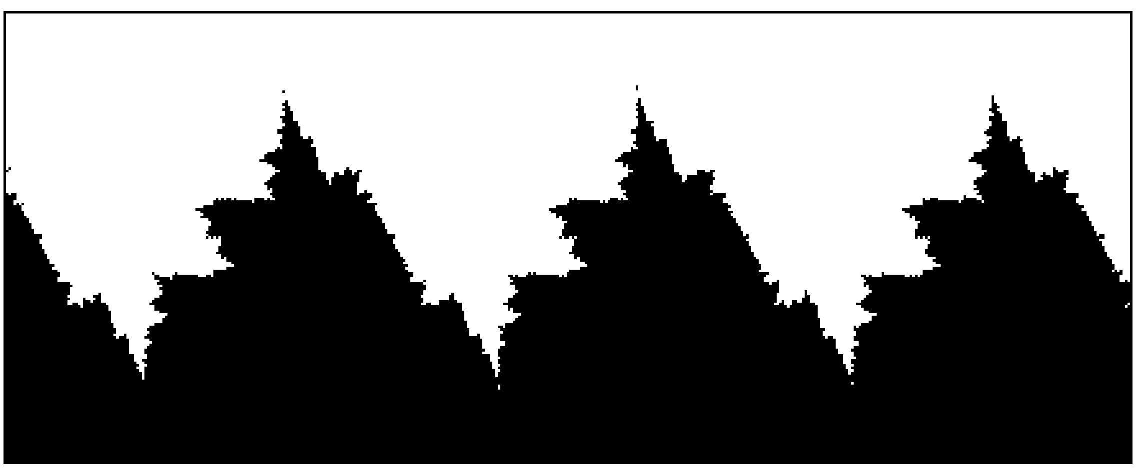

FIGURE 5. The parabolic basins of the function from Example 5. The range shown is $|\operatorname{Re} z| < 5, |\operatorname{Im} z| < 2$.

where T is the Nevanlinna characteristic. Langley [17, 18] proved that meromorphic functions with the property $T(r, f) = o((\log r)^2)$ have infinitely many singular values.

Let h be the branch of arccosine arccos which maps the 4-th quadrant $Q_4 = \{z : \operatorname{Re} z > 0, \operatorname{Im} z < 0\}$ onto the half-strip $H = \{z : \operatorname{Re} z \in (0, \pi/2), \operatorname{Im} z > 0\}$.

Let g be the conformal map of a rectangle $R = \{z : \operatorname{Re} z \in (0, \pi/2), \operatorname{Im} z \in (0, a)\}$ with $a > 0$ onto Q_4, such that $g(\pi/2) = 0$ and $g(\pi/2 + ia) = \infty$, and $g(ia) > g(0) > 0$. By the Reflection Principle, g has an analytic continuation to

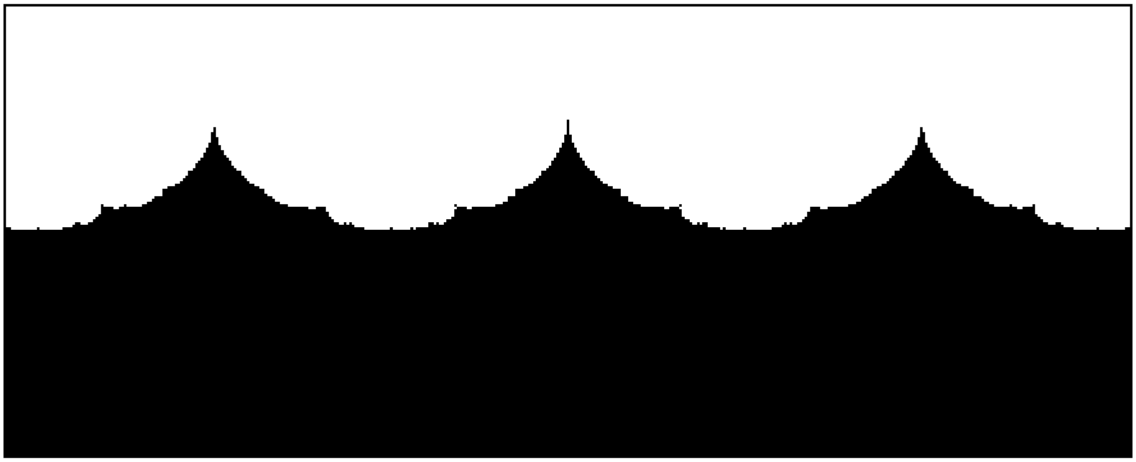

FIGURE 6. The parabolic basin of the function from Example 6 is shown in white, the attracting one in black. The range shown is $|\operatorname{Re} z| < 5, |\operatorname{Im} z| < 2$.

the half-strip H and maps this half-strip into the left half-plane. It is easy to see that g is an elliptic function.

The composite function $f = g \circ h$ maps the positive ray into itself, and applying the reflection again we conclude that it maps the right half-plane into itself. The boundary values on the imaginary axis belong to the imaginary axis, so by another reflection f extends to a meromorphic function in the plane. We see that both the right and left half-plane are completely invariant.

The function f has 4 critical values, $\pm g(ia)$ and $\pm g(0)$, two in the right half-plane and two in the left half-plane.

To estimate the growth of f is it enough to notice that $\arccos z = i \log z + O(1)$ as $z \to \infty$ in the lower half-plane and in the upper half-plane. Taking into account that g is an elliptic function we obtain (7).

Our function f satisfies the differential equation

$$(1 - z^2)(f')^2 = c(f^2 - p^2)(f^2 - q^2),$$

where $p = g(ia)$, $q = g(0)$ and c is a real constant.

A similar differential equation was considered by Bank and Kaufman [6]; see also [16, 19].

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