Sums of reciprocal powers and Bernoulli numbers

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To find the sum

$$\zeta(2m) = \sum_{n=1}^{\infty} n^{-2m}$$

we write it as

$$\zeta(2m) = \frac{1}{2} {\sum}' n^{-2m},$$

where $\sum'$ means summation over all non-zero integers, positive and negative. Then we apply residue theorem to the function

$$f(z) = z^{-2m} \pi \cot \pi z$$

in the large rectangle $\{x + iy : -M \leq x \leq M, \ -M \leq y \leq M\}$, where $M \in \mathbf{Z} + 1/2$ is a half-integer. It is easy to see that the integral over the boundary of this rectangle tends to 0 as $M \to \infty$, so the sum of the residues at all integers is 0. Snce the residue of $z^{-2m}\pi \cot \pi z$ at a non-zero integer n equals n^{-2m}, we obtained the formula

$$\zeta(2m) = \frac{1}{2} {\sum}' \frac{1}{n^{2m}} = -\frac{1}{2} \mathrm{res}_0 \pi z^{-2m} \cot \pi z. \tag{1}$$

To obtain this residue one needs Taylor coefficients of cot. Let us begin with a simpler function and define B_n by the formula

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_n}{n!} z^n.$$

One can find these B_n reccursively:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \sum_{n=1}^{\infty} \frac{z^n}{n!} = z.$$

We obtain:

$$B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6, \quad B_3 = 0, \quad B_4 = -1/30,$$

and so on. The pattern is not clearly visible, but it is evident that these numbers are rational. Let us prove that all odd numbered B_n, except B_1 are zero.

This means that the function

$$\frac{z}{e^z - 1} + \frac{z}{2}$$

is even. This is a rare example when this property is not seen immediately, but anyway, this is a routine verification. We will later see that B_{2n} switch signs. They are called Bernoulli numbers.

We will express Taylor coefficients of $\cot z$ in terms of Bernoulli numbers.

$$\begin{aligned} z \cot z &= iz \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = iz \frac{e^{2iz} + 1}{e^{2iz} - 1} \\ &= iz + \frac{2iz}{e^{2iz} - 1} = iz + \sum_{n=0}^{\infty} \frac{B_n}{n!} (2iz)^n. \end{aligned}$$

We found that $B_1 = -1/2$, so the term with z cancels, and the sum of even powers remains. Substituting πz we obtain:

$$\pi z \cot \pi z = \sum_{n=0}^{\infty} (-1)^n \frac{B_{2n}}{(2n)!} (2\pi z)^{2n}.$$

The residue in (1) is the constant term of $z^{-2m} \pi z \cot \pi z$. So we obtain

$$\sum_{n=1}^{\infty} n^{-2m} = (-1)^{m-1} \frac{B_{2m}}{(2m)!} (\pi)^{2m} 2^{2m-1}.$$

As the right hand side is evidently positive, we conclude that Bernoulli numbers have a sign switch.

Bernoulli numbers occur in many places in mathematics, see, for example https://mathoverflow.net/questions/61252.