Roots of polynomials with positive coefficients

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Preliminary version.

Abstract

We describe the limit zero distributions of sequences of polynomials with positive coefficients. We also characterize the polynomials with real coefficients for which some power has positive coefficients.

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1 Introduction and results

If f is a polynomial with non-negative coefficients, then evidently

$$|f(z)| \leq f(|z|), \quad z \in \mathbf{C}. \tag{1.1}$$

The converse is not true. Linnik and Ostrovskii [8, p. 32] give the simple example

$$f(z) = 1 + 2z - z^2 + 3z^3 + 3z^4,$$

which satisfies (1.1) because f^2 has positive coefficients.

Our first result is

Theorem 1. *Let*

$$f(z) = a_0 + \ldots + a_d z^d, \quad a_0 > 0,\ a_d > 0, \tag{1.2}$$

be a real polynomial. The following conditions are equivalent:

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(*i*) *There exists a positive integer m such that all coefficients of f^m are strictly positive.*

(*ii*) *There exists a positive integer m_0 such that for all $m \geq m_0$, all coefficients of f^m are strictly positive.*

(*iii*) *The inequalities*

$$|f(z)| < f(|z|), \quad z \notin [0, \infty), \tag{1.3}$$

and

$$a_1 > 0, \quad a_{d-1} > 0 \tag{1.4}$$

hold.

Corollary 1. *Let g be an entire function with non-negative coefficients and let f be a real polynomial satisfying condition* (*iii*) *of Theorem 1. Then all but finitely many coefficients of $g \circ f$ are non-negative.*

In the case that $g(z) = e^z$, a much stronger result was obtained by Hayman [5, Theorem X]. He actually obtained for a real polynomial f a necessary and sufficient condition for all but finitely many coefficients of $\exp f$ to be positive.

Theorem 1 will allow us to answer the following question asked by Ofer Zeitouni and Subhro Ghosh [13].

Let P be a polynomial. Consider the discrete probability measure $\mu[P]$ in the plane which has an atom of mass $m/\deg P$ at every zero of P of multiplicity m. It is called the "empirical measure" in the theory of random polynomials.

Let μ_n be a sequence of empirical measures of some polynomials with positive coefficients, and suppose that $\mu_n \to \mu$ weakly. The question is how to characterize all possible limit measures μ. We give such a characterization in terms of logarithmic potentials.

Theorem 2. *For a measure μ to be a limit of empirical measures of polynomials with positive coefficients, it is necessary and sufficient that the following conditions are satisfied:*

μ is symmetric with respect to the complex conjugation, $\mu(\mathbf{C}) \leq 1$, and the potential

$$u(z) = \int_{|\zeta| \leq 1} \log |z - \zeta| d\mu(\zeta) + \int_{|\zeta| > 1} \log \left| 1 - \frac{z}{\zeta} \right| d\mu(\zeta) \tag{1.5}$$

has the property

$$u(z) \le u(|z|). \tag{1.6}$$

The potential in Theorem 2 converges for every positive measure with the property $\mu(\mathbf{C}) < \infty$ to a subharmonic function $u \not\equiv -\infty$. If

$$\int_{|\zeta|>1} \log|\zeta| d\mu(\zeta) < \infty \quad \text{or} \quad \int_{|\zeta|<1} \log \frac{1}{|\zeta|} d\mu(\zeta) < \infty,$$

then the definition of u in Theorem 2 can be simplified to

$$\int_{\mathbf{C}} \log|z-\zeta| d\mu(\zeta) \quad \text{or} \quad \int_{\mathbf{C}} \log\left|1 - \frac{z}{\zeta}\right| d\mu(\zeta),$$

respectively. When both integrals exist, all three potentials differ from each other by additive constants.

Obrechkoff [9] proved that empirical measures of polynomials with non-negative coefficients satisfy

$$\mu(\{z \in \mathbf{C}^* \colon |\arg z| \le \alpha\}) \le \frac{2\alpha}{\pi} \mu(\mathbf{C}^*), \quad 0 \le \alpha \le \pi/2. \tag{1.7}$$

We call this the Obrechkoff inequality. The limits of these measures also satisfy (1.7).

Combining our result with Obrechkoff's theorem we conclude that (1.6) and symmetry of the measure imply (1.7). In particular we find that Obreschkoff's inequality is satisfied not only by polynomials with non-negative coefficients, but more generally by polynomials satisfying (1.1).

The converse does not hold; that is, the inequalities (1.1) and (1.6) do not follow from Obrechkoff's inequality. Indeed, let

$$P(z) = (z^2+1)^m (z^2 - 2z\cos\beta + 1)$$

This polynomial has roots of multiplicity m at $\pm i$, and simple roots at $\exp(\pm i\beta)$. Obrechkoff's inequality is satisfied if $\beta \ge \pi/(2m+2)$. On the other hand, $P(1) < |P(-1)|$ for all m and $\beta \in (0, \pi/2)$.

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2 Proof of Theorem 1

It is evident that $(ii) \Rightarrow (i) \Rightarrow (iii)$. So it remains to prove that $(iii) \Rightarrow (ii)$. So suppose that (iii) holds.

The first coefficients of the polynomials f^m will be estimated by the following two propositions. These results hold not only for polynomials, but more generally for power series. We can restrict to the case that $f(0) = 1$. So let

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \quad \text{and} \quad f(x)^m = 1 + \sum_{n=1}^{\infty} a_n^{(m)} x^n. \tag{2.1}$$

Proposition 1. *Let f be a formal power series given by* (2.1)*. Suppose that $a_1 > 0$. Then for every $N \in \mathbf{N}$ there exists $M \in \mathbf{N}$ such that if $1 \leq n \leq N$ and $m \geq M$, then $a_n^{(m)} > 0$.*

Proof. Without loss of generality we assume that $a_1 = 1$. This can be achieved by a scaling of the independent variable. Then

$$a_n^{(m)} \geq \binom{m}{n} - p_n m^{n-1} A_n,$$

where p_n is the number of partitions of n, and A_n depends on the first n coefficients of the series. As the first term has degree n, with respect to m, it dominates when m is large enough. □

Proposition 2. *Let f be a power series given by* (2.1) *with positive radius of convergence. Suppose that $a_1 > 0$. Then there exist $\delta > 0$ and $M \in \mathbf{N}$ such that if $m \geq M$ and $1 \leq n \leq \delta m$, then $a_n^{(m)} > 0$.*

Proof. The proof is based on the saddle point method. We assume without loss of generality that $a_1 = 1$. Noting that

$$t\frac{f'(t)}{f(t)} = t + O(t^2) \quad \text{as } t \to 0, \tag{2.2}$$

we see that there exists $t_0 > 0$ such that $tf'(t)/f(t)$ is increasing in the interval $[0, t_0]$. We deduce that there exists $\delta > 0$ such that if $x \in (0, \delta]$, then there exists a unique $r \in (0, t_0]$ such that

$$r\frac{f'(r)}{f(r)} = x. \tag{2.3}$$

We apply this for $x = n/m$. We will always assume that $0 < x \le \delta$, with δ so small that various conditions imposed later are also satisfied. All these conditions will depend only on f. With

$$\phi(z) = \log f(z) - x \log z \quad \text{and} \quad h(\theta) = \phi(re^{i\theta})$$

we then have

$$2\pi a_n^{(m)} = \int_{|z|=r} \frac{f(z)^m}{z^n} \frac{dz}{iz} = \int_{-\pi}^{\pi} e^{m\phi(re^{i\theta})} d\theta = \int_{-\pi}^{\pi} e^{mh(\theta)} d\theta. \tag{2.4}$$

We have

$$\phi'(z) = \frac{f'(z)}{f(z)} - \frac{x}{z}, \quad \text{and} \quad h'(\theta) = ire^{i\theta}\phi'(re^{i\theta}).$$

Our choice of r in (2.3), and this is the essential point in the saddle point method, yields that

$$\phi'(r) = 0 \quad \text{and} \quad h'(0) = 0. \tag{2.5}$$

Next we note that

$$\phi''(z) = \frac{f''(z)}{f(z)} - \left(\frac{f'(z)}{f(z)}\right)^2 + \frac{x}{z^2} =: A(z) + \frac{x}{z^2}, \tag{2.6}$$

and

$$\phi'''(z) = \frac{f'''(z)}{f(z)} - 3\frac{f''(z)f'(z)}{f(z)^2} + 2\left(\frac{f'(z)}{f(z)}\right)^3 - \frac{2x}{z^3} =: B(z) - 2\frac{x}{z^3}, \tag{2.7}$$

where A and B are bounded in some neighborhood of 0. With $z = re^{i\theta}$ we have

$$h''(\theta) = -z^2\phi''(z) - z\phi'(z) \tag{2.8}$$

and

$$h'''(\theta) = -iz^3\phi'''(z) - 3iz^2\phi''(z) - iz\phi'(z). \tag{2.9}$$

Noting that

$$x \sim r \quad \text{as } x \to 0, \tag{2.10}$$

by (2.2) and (2.3) we obtain

$$|z\phi'(z)| = \left| z\frac{f'(z)}{f(z)} - x \right| \le r + x + O(r^2) = 2x + O(x^2) \quad \text{for } |z| = r$$

as $x \longrightarrow 0$, and thus $|z\phi'(z)| \le 3x$ for $|z| = r$ if δ is sufficiently small. Similarly, $|z^2\phi''(z)| \le 2x$ and $|z^3\phi'''(z)| \le 3x$ for for $|z| = r$ and small δ by (2.6) and (2.7). Thus

$$|h'''(\theta)| \le 12x \quad \text{for } -\pi \le \theta \le \pi \tag{2.11}$$

by (2.9), provided δ is small. Thus we have the expansion

$$h(\theta) = h(0) - \tau\theta^2 + R(\theta) \quad \text{with} \quad |R(\theta)| \le 2x|\theta|^3, \tag{2.12}$$

where

$$\tau = -\frac{1}{2}h''(0) = \frac{1}{2}r^2\phi''(r) \tag{2.13}$$

by (2.5) and (2.8). From (2.6) and (2.10) we deduce that

$$\frac{1}{4}x \le \tau \le \frac{3}{4}x \tag{2.14}$$

for small δ.

We choose $N = 10^9$ and note that

$$8\pi\sqrt{n}\exp\left\{-\frac{n^{1/3}}{32}\right\} < 1 \quad \text{for } n \ge N. \tag{2.15}$$

Using Proposition 1 we conclude that the first N coefficients of f^m are positive for large m, so it is sufficient to restrict our attention to the coefficients $a_n^{(m)}$ with $n \ge N$.

We put

$$\theta_n = \frac{1}{2}n^{-1/3}, \tag{2.16}$$

and split the integral in (2.4) into two parts:

$$I_1 = \int_{-\theta_n}^{\theta_n} e^{mh(\theta)}d\theta \quad \text{and} \quad I_2 = \int_{\theta_n \le |\theta| \le \pi} e^{mh(\theta)}d\theta. \tag{2.17}$$

For $|\theta| \le \theta_n$ we deduce from (2.12) that

$$|\operatorname{Im}(mh(\theta))| = |\operatorname{Im}(mR(\theta))| \le m|R(\theta)| \le 2mx\theta_n^3 = 2n\theta_n^3 = \frac{1}{4} \le \frac{\pi}{3},$$

and also

$$m|R(\theta)| \le 2mx\theta_n\theta^2 = \frac{1}{n^{1/3}}mx\theta^2 \le \frac{1}{N^{1/3}}mx\theta^2 \le m\tau\theta^2,$$

so that

$$\operatorname{Re} mh(\theta) \geq mh(0) - 2m\tau\theta^2.$$

Thus

$$\operatorname{Re} e^{mh(\theta)} = \cos(\operatorname{Im}(mh(\theta)))e^{\operatorname{Re} mh(\theta)} \geq \frac{1}{2}e^{mh(0)}e^{-2m\tau\theta^2} \tag{2.18}$$

for $|\theta| \leq \theta_n$, and hence

$$\operatorname{Re} I_1 \geq \frac{1}{2}e^{mh(0)} \int_{\theta_n}^{\theta_n} e^{-2m\tau\theta^2} d\theta = \frac{e^{mh(0)}}{2\sqrt{2m\tau}} \int_{\sqrt{2m\tau}\theta_n}^{\sqrt{2m\tau}\theta_n} e^{-t^2} dt.$$

As

$$\sqrt{2m\tau}\theta_n \geq \sqrt{\frac{mx}{2}}\theta_n = \frac{n^{1/6}}{2\sqrt{2}} \geq \frac{10^{3/2}}{2\sqrt{2}} = 5\sqrt{5} \geq 10,$$

and also $2\sqrt{2m\tau} \leq 2\sqrt{3mx/2} = 2\sqrt{3n/2} = \sqrt{6n}$, we conclude that

$$\operatorname{Re} I_1 \geq \frac{e^{mh(0)}}{\sqrt{6n}} \int_{-10}^{10} e^{-t^2} dt \geq \frac{1}{2}\frac{e^{mh(0)}}{\sqrt{n}}. \tag{2.19}$$

Now we estimate $\operatorname{Re} I_2$ from above. For $|\theta| \leq 1/16$ we have

$$|R(\theta)| \leq 2x|\theta|^3 \leq \frac{1}{8}x\theta^2 \leq \frac{1}{2}\tau\theta^2,$$

and thus

$$\operatorname{Re} h(\theta) \leq h(0) - \frac{1}{2}\tau\theta^2 \leq h(0) - \frac{1}{8}x\theta^2,$$

by (2.12) and (2.14). For $\theta_n \leq |\theta| \leq 1/16$ this yields

$$\operatorname{Re} mh(\theta) \leq mh(0) - \frac{1}{8}mx\theta_n^2 = mh(0) - \frac{1}{32}n^{1/3}. \tag{2.20}$$

Next we show that the last inequality also holds for $1/16 \leq |\theta| \leq \pi$, provided that δ is sufficiently small. In fact, we have $\log f(z) = z + O(z^2)$ as $z \to 0$ and thus

$$\operatorname{Re} h(\theta) - h(0) = \operatorname{Re} \log f(re^{i\theta}) - \log f(r) = r(\cos\theta - 1) + O(r^2).$$

As $1 - \cos(1/16) \geq 10^{-3}$ this yields for small δ that

$$\operatorname{Re} h(\theta) - h(0) \leq -10^{-3}x$$

and hence

$$\operatorname{Re} mh(\theta) - mh(0) \leq -10^{-3}mx = -10^{-3}n \leq -10^{-3}N^{2/3}n^{1/3} = -10^{3}n^{1/3}$$

for $|\theta| \geq 1/16$. Thus (2.20) holds for $\theta_n \leq |\theta| \leq \pi$ so that

$$\operatorname{Re} I_2 \leq 2\pi e^{mh(0)} \exp\left(-\frac{1}{32}n^{1/3}\right).$$

Using (2.15) we see that $\operatorname{Re} I_2 \leq \operatorname{Re} I_1/2$. Now we deduce from (2.4) and (2.17) that $a_n^{(m)} > 0$. □

The proof of Theorem 1 also requires the following result.

Lemma 1. *Let u be a harmonic function in a neighborhood of a point $r > 0$, and suppose that u satisfies* (1.6). *Then*

$$\left(\frac{d}{d\log r}\right)^2 u(r) > 0,$$

unless $u(z) \equiv a \log |z| + b$ with some real constants a and b.

Proof. Put $v(\zeta) = u(re^{\zeta})$. Then v is harmonic in a neighborhood of 0, and (1.6) becomes

$$v(x + iy) \leq v(x). \tag{2.21}$$

This implies that $v_{yy}(0) \leq 0$. Hence $v_{xx}(0) \geq 0$ since v is harmonic. Thus the statement of the Lemma is equivalent to $v_{xx}(0) \neq 0$.

Subtracting a constant from v does not change condition (2.21), so we may assume in addition that $v(0) = 0$. Consider the Taylor expansion of v at 0. In view of (2.21), the first degree term is ax. We subtract this term from v without altering (2.21), so that now $v(\zeta) = O(\zeta^2), \quad \zeta \to 0$.

Proving the Lemma by contradiction, suppose that $v_{xx}(0) = 0$. Then $v_{yy} = 0$, because the function is harmonic, and (2.21) implies $v_{xy} = 0$. Thus all second degree terms in the Taylor expansion vanish.

Let w be the lowest degree homogeneous polynomial in the Taylor expansion of v. Then

$$w(\rho e^{it}) = c\rho^m \cos(mt), \tag{2.22}$$

where $m \geq 3$ and $c \neq 0$, unless v is linear, that is $u(z) = a \log |z| + b$. But (2.22) is incompatible with (2.21). Indeed at the point $\rho \exp(2\pi i/m)$ the right hand side of (2.22) is equal to ρ^m while at the point $\rho \cos(2\pi/m)$ it is equal to $\rho^m \cos^m(2\pi/m) < \rho^m$, so for sufficiently small ρ the inequality (2.21) is violated. □

Completion of the proof of Theorem 1. We may assume without loss of generality that $f(0) = 1$. Proposition 2 shows that the coefficients $a_n^{(m)}$ of f^m are positive for $m \geq M$ and $n/m \leq \delta$. Applying Proposition 2 to the reverse polynomial

$$z^{-d} f(1/z)$$

we obtain the positivity of the coefficients $a_n^{(m)}$ for $n/m \geq m(d-\delta)$. Thus it remains to prove the conclusion for $x = n/m \in [\delta, d-\delta]$.

We use the same saddle point argument as in the proof of Proposition 2, but now our reasoning is simpler. We use the same notation as there. In particular, we choose r as the unique solution of (2.3). We note here that $f(r) = M(r, f)$ by (1.3) so that

$$r\frac{f'(r)}{f(r)} = \frac{d \log M(r, f)}{\log r}$$

is an increasing function of r by the Hadamard Three Circles Theorem, but we actually have

$$h''(0) = -r^2\phi''(r) = -\frac{d}{d\log r}\left(r\frac{f'(r)}{f(r)}\right) < 0 \tag{2.23}$$

by Lemma 1. The exceptional function in Lemma 1 does not satisfy (1.3). The condition $x = n/m \in [\delta, d-\delta]$ corresponds to $r \in [r_0, R_0]$ for suitable values r_0 and R_0 depending only on f.

We again have (2.12) and (2.13), except that in (2.12) we only have the estimate $|R(\theta)| \leq c_1|\theta|^3$ with some constant c_1. Instead of (2.14) we now deduce from (2.23) that $c_2 \leq \tau \leq c_3$ for certain positive constants c_2 and c_3.

We use

$$\theta_m = m^{-3/8} \tag{2.24}$$

instead of (2.16) and split the integral into two parts

$$I_1 = \int_{-\theta_m}^{\theta_m} e^{mh(\theta)} d\theta \quad \text{and} \quad I_2 = \int_{\theta_m \leq |\theta| \leq \pi} e^{mh(\theta)} d\theta.$$

For $|\theta| \leq \theta_m$ and large m we again find that

$$|\operatorname{Im}(mh(\theta))| \leq m|R(\theta)| \leq c_1 m\theta_m^3 = c_1 m^{-1/8} \leq \frac{\pi}{3}$$

and

$$m|R(\theta)| \le c_1 m \theta_m \theta^2 \le c_2 m \theta^2 \le m\tau\theta^2.$$

As before this yields (2.18) for $|\theta| \le \theta_m$ and hence

$$\operatorname{Re} I_1 \ge \frac{1}{2} e^{mh(0)} \int_{\theta_m}^{\theta_m} e^{-2m\tau\theta^2} d\theta \ge \frac{e^{mh(0)}}{2\sqrt{2m\tau}} \int_{-10}^{10} e^{-t^2} dt \ge \frac{e^{mh(0)}}{\sqrt{m\tau}} \ge \frac{e^{mh(0)}}{\sqrt{c_3 m}}.$$

For $\theta_m \le |\theta| \le c_2/(2c_1)$ we have

$$|R(\theta)| \le \frac{c_2}{2}\theta^2 \le \frac{\tau}{2}\theta^2$$

and thus

$$\operatorname{Re} mh(\theta) \le mh(0) - \frac{m\tau}{2}\theta_m^2 \le mh(0) - \frac{c_2}{2} m^{1/4}. \tag{2.25}$$

By hypothesis (1.3), and since $r \in [r_0, R_0]$, there exists $\varepsilon > 0$ such that

$$\operatorname{Re} h(\theta) \le h(0) - \varepsilon$$

for $c_2/(2c_1) \le |\theta| \le \pi$. Thus (2.25) also holds for θ in this range, provided m is large. Hence

$$\operatorname{Re} I_2 \le 2\pi e^{mh(0)} \exp\left(-\frac{c_2}{2} m^{1/4}\right).$$

For large m we again find that $\operatorname{Re} I_2 < \operatorname{Re} I_1$ and thus $a_n^{(m)} > 0$. □

3 Examples and remarks

It follows from Theorem 1 that if P is a polynomial of degree at most 3, then P has positive coefficients if and only if (1.3) is satisfied. The first non-trivial case occurs when $\deg P = 4$, the first two and the last two coefficients are positive while the middle one is negative. Therefore we consider the polynomial

$$P_c(z) = 1 + z + cz^2 + z^3 + z^4.$$

Proposition 3.
a) P_c *satisfies* (1.1) *if and only if* $c \ge -7/8$.
b) P_c *satisfies* (1.3) *if and only if* $c > 7/8$.

Proof. Instead of P_c we consider the Laurent polynomial

$$z^{-2}P_c(z) = z^{-2} + z^{-1} + c + z + z^2 = (w + 1/2)^2 - 5/4 - c,$$

where

$$w(z) = z + z^{-1}.$$

Function $w(z)$ maps the unit circle onto the interval $[-2, 2]$, and the circles $|z| = r$ onto ellipses whose horizontal axes are longer than vertical axes. This property persists when we shift such an ellipse horizontally. So the maximum of $|\zeta^2 - 5/4 - c|$ on such a shifted ellipse can occur only at its intersection with the real line. We conclude that the statements a) and b) hold.

If f is as in (2.1) with $a_1 = 0$, then clearly $a_1^{(m)} = 0$ for all m. The following example shows that if the hypothesis (1.4) is omitted in Theorem 1, then every power of f may actually have some negative coefficients.

Example 1. Let

$$f(z) = 1 + z^3 + z^4 - az^5 + z^6 + z^7 + z^{10}.$$

It is easy to see that a is a sufficiently small positive number, then f satisfies (1.3). However, we have $a_5^{(m)} = -ma < 0$ and $a_{10m-5}^{(m)} = -ma < 0$ for all m.

It is an interesting problem to describe the polynomials with the property that their sufficiently high powers have non-negative coefficients. Such polynomials must have property (1.3) unless they are of the form $z^p g(z^q)$ for some non-negative integers p and $q \geq 2$. We conjecture that for polynomials of the form

$$f(z) = a_0 + a_2 z^2 + a_3 z^3 + \ldots + a_{d-3} z^{d-3} + a_{d-2} z^{d-2} + a_d z^d,$$

with $a_0 > 0, a_2 > 0, a_3 > 0, a_{d-3} > 0, a_{d-2} > 0, a_d > 0$, and satisfying (1.3), all sufficiently high powers f^m have non-negative coefficients.

Now we show that Theorem 1 may fail for entire functions.

Example 2. Let $Q(z) = a_0 + \ldots + a_d z^d$ be any polynomial with $a_0, a_1 \neq 0$ that satisfies (1.1) but does not satisfy (1.3), for example, $Q = P_{-7/8}$. Then all positive powers of Q have some negative coefficients.

Now define

$$f(z) = 1 + z + Q(z)\sum_{j=1}^{\infty} \frac{z^{n_j}}{n_j!},$$

where n_j is any sequence with the property $n_{j+1}/n_j \to \infty$. Then f satisfies (1.3) but it is easy to see that every power has some negative coefficients.

Remark 1. In [2, p. 209], [3] and various other papers, D'Angelo studies the Cauchy-Schwarz type condition

$$|r(z,\overline{w})|^2 \le r(z,\overline{z})r(w,\overline{w}) \tag{3.1}$$

for polynomials $r\colon \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$. For $r(z,w) = f(zw)$ with a polynomial $f\colon \mathbf{C} \to \mathbf{C}$ this condition takes the form

$$|f(z\overline{w})|^2 \le f(|z|^2)f(|w|^2) \quad \text{for all } z,w \in \mathbf{C}. \tag{3.2}$$

It turns out that this condition is equivalent to the condition (1.1) that we considered.

Indeed, setting $\overline{w} = z$ in (3.2) we obtain (1.1) with z replaced by z^2. On the other hand, if (1.1) holds, then

$$|f(z\overline{w})|^2 \le f(|z||w|)^2 \le f(|z|^2)f(|w|^2),$$

where the second inequality holds by the Hadamard Three Circles Theorem.

Similarly, the strict inequality

$$|f(z\overline{w})|^2 < f(|z|^2)f(|w|^2) \quad \text{for } z \ne w \tag{3.3}$$

is equivalent to the strict inequality (1.3) occurring in Theorem 1, (iii). The proof is the same, using that equality in the Hadamard Three Circles Theorem occurs only for monomials. Alternatively, one may use the stronger convexity property given by Lemma 1.

Identifying a polynomial with the vector of coefficients, we define $K_d \subset \mathbf{R}^{d+1}$ as the set of all polynomials of degree at most d which satisfy (1.1) or, equivalently, which satisfy (3.2). This is a closed cone in $\mathbf{R}^{d+1}$, which is also closed with respect to multiplication.

The subset of K_d consisting of polynomials satisfying (1.3) is not open in $\mathbf{R}^{d+1}$, but the interior $\text{int}(K)$ of K_d consists of all polynomials of degree d with $a_0 > 0$ and $a_d > 0$ which satisfy (1.3) and (1.4). So condition (iii) of

Theorem 1 can be restated as $f \in \mathrm{int}(K_d)$. Now it can be deduced from the first inclusion of [3, Theorem 7.1] that if $f \in \mathrm{int}(K_d)$, then there exists $m \in \mathbf{N}$ such that the coefficients of f^m are non-negative. Noting that $f \in \mathrm{int}(K_d)$ is an open condition, one obtains in fact that the coefficients of f^m are positive. With these observations one can thus deduce the implication $(iii) \Rightarrow (i)$ in Theorem 1 from [3, Theorem 7.1].

We mention that the inclusion quoted is not proved in [3], but it is stated that it can be derived from [1] and [12].

Our methods are completely different from those in [1, 3, 12]. Moreover, they yield additional results such as Propositions 1 and 2, as well as those mentioned in the following remark.

Remark 2. Our method allows to prove the positivity of certain coefficients of f^m even if f does not satisfy (1.3). We illustrate this by an example. Let $Q = P_{-7/8}$ be the polynomial already considered above and write

$$Q(x)^m = \sum_{n=0}^{4m} a_n^{(m)} x^n.$$

The arguments in Example 1 show that $|Q(z)| < Q(|z|)$ if $z \notin [0, \infty)$ and $|z| \neq 1$. In other words, the condition (1.3) is violated only at certain points on the unit circle. We define x by (2.3) with $r = 1$, that is,

$$x = \frac{Q'(1)}{Q(1)} = 2.$$

Our proof shows that given $\varepsilon > 0$, there exists $m_0 \in \mathbf{N}$ such that $a_n^{(m)} > 0$ whenever $m \geq m_0$ and $|n/m - 2| \geq \varepsilon$.

Quite generally, let f be a polynomial of degree d with real coefficients and write

$$f(x)^m = \sum_{n=0}^{dm} a_n^{(m)} x^n.$$

Suppose that $|f(z)| < f(|z|)$ for $r_1 \leq |z| \leq r_2$ and $z \notin [r_1, r_2]$, and put $x_j = r_j f'(x_j)/f(x_j)$ for $j = 1, 2$. Then there exists $m_0 \in \mathbf{N}$ such that $a_n^{(m)} > 0$ whenever $m \geq m_0$ and $n/m \in [x_1, x_2]$.

Remark 3. Polynomials with positive coefficients are important as generating functions of bounded random variables whose values are non-negative integers. In general, for a probability measure ν on the real line, one defines the characteristic function

$$F(z)=\int_{\mathbf{R}} e^{-itz}d\nu(t). \tag{3.4}$$

If the measure decays sufficiently fast at infinity, the characteristic function is analytic in a horizontal strip $a<\operatorname{Im} z<b$ where $a\leq 0\leq b$. Characteristic functions have the property

$$|F(x+iy)|\leq F(iy),\quad y\in(a,b),\quad x\in\mathbf{R}, \tag{3.5}$$

which is analogous to (1.1). Functions satisfying (3.5) are called ridge functions. When the measure ν is discrete and has finitely many atoms b_j at non-negative integers j, then

$$F(z)=\sum_{j=0}^{d} b_j e^{-ijz}=P(e^{-iz}),$$

where P is a polynomial with positive coefficients.

Analytic ridge functions and their relation to analytic characteristic functions were studied much, see, for example, [4, 10, 11]. It is interesting, to what extent our results can be generalized to this case.

4 Characterization of limit measures

In this section we use some facts about subharmonic functions and potential theory which can be found in [7]. We recall that the Riesz measure of a subharmonic function u is $(2\pi)^{-1}\Delta u$, where the Laplacian is understood as a Schwartz distribution. In particular the empirical measure of a polynomial P of degree d is the Riesz measure of the subharmonic function $(\log|P|)/d$. For the general properties of convergence of subharmonic functions we refer to [7, Theorem 3.2.13]. This result will be used repeatedly and is stated for the convenience of the reader as Theorem A in the Appendix.

Proof of Theorem 2. For a subharmonic function u we put

$$B(r,u)=\max_{|z|\leq r} u(z)$$

and notice that condition (1.6) can be rewritten as

$$B(r,u) = u(r), \quad r \geq 0, \tag{4.1}$$

in view of the Maximum Principle. This implies that $u(r)$ is strictly increasing for non-constant subharmonic functions u satisfying (1.6). Moreover, the Hadamard Three Circles Theorem implies that $u(r) = B(r,v)$ is convex with respect to $\log r$, so $u(r)$ is continuous for $r > 0$.

First we prove the necessity of our conditions. Let f_n be a sequence of polynomials with non-negative coefficients. Then $u_n = \log|f_n|/\deg f_n$ are subharmonic functions whose Riesz measures μ_n are the empirical measures of f_n. As the μ_n are probability measures, every sequence contains a subsequence for which the weak limit μ exists. This μ evidently satisfies $\mu(\mathbf{C}) \leq 1$, and μ is symmetric with respect to complex conjugation. Consider the potential u defined by (1.5). This is a subharmonic function, $u \not\equiv -\infty$, and we have $u_n + c_n \to u$ for suitable constants c_n.

For a complete discussion of the mode of convergence here we refer to the Appendix; what we need is that $u_n(r) + c_n \to u(r)$ at every point $r > 0$ and for all other points

$$\limsup_{n\to\infty} u_n(z) + c_n \leq u(|z|).$$

As the polynomials f_n have non-negative coefficients, they satisfy (1.1), and the u_n satisfy (1.6). Thus u satisfies (1.6).

In the rest of this section we prove sufficiency. We start with a measure μ such that the associated potential u in (1.5) satisfies (1.6) and

$$u(z) = u(\overline{z}). \tag{4.2}$$

The idea is to approximate u by potentials of the form $(\log|f_n|)/\deg f_n$, where the f_n are polynomials with real coefficients that satisfy the assumptions of Theorem 1. Applying Theorem 1 we find that f_n^m has positive coefficients for some m. But f_n^m has the same empirical measure as f_n, which is close to μ.

If $u(z) = k\log|z|$, then we approximate u with

$$u_n(z) = k_n \log|z| + (1-k_n)\log|z+n|,$$

where k_n is a sequence of rational numbers such that $k_n \to k$, $0 \leq k_n \leq 1$. For the rest of the proof we assume that $u(z)$ is not of the form $k\log|z|$.

The approximation of u will be performed in several steps. In each step we modify the function obtained on the previous step, and starting with u obtain subharmonic functions $u_1, \dots, u_5$. Each modification will preserve the asymptotic inequality

$$u(z) \leq O(\log |z|), \quad z \to \infty.$$

It is well known that every subharmonic function in the plane which satisfies this inequality can be represented in the form (1.5) plus a constant, and we will call functions of this form simply "potentials", see, for example [6, Thm. 4.2] (case $q = 0$).

1. Fix $\varepsilon > 0$ and define

$$u_1(z) = \max\{u(ze^{i\alpha})\colon |\alpha| \leq \varepsilon\}.$$

It is easy to see that u is the potential of some finite measure, and that $u_1 \to u$ when $\varepsilon \to 0$. This implies that the Riesz measure of u_1 is close (in the weak topology) to that of u.

Evidently, u_1 satisfies (1.6) and (4.2), and $u_1(re^{i\theta}) = u(r)$ for $|\theta| \leq \varepsilon$. Thus $u_1(re^{i\theta}) = u(r)$ does not depend on θ for $|\theta| \leq \varepsilon$.

2. Choose $\delta \in (0, \varepsilon)$ and consider the solution v of the Dirichlet problem in the sector

$$D = \{z\colon |\arg z| < \delta\}$$

with boundary conditions $u_1(z)$ and satisfying $v(z) = O(\log |z|)$ as $z \to \infty$. To prove the existence and uniqueness of v, we map D conformally onto the upper half-plane, and apply Poisson's formula to solve the Dirichlet problem. The growth restriction near ∞ ensures that the solution of the Dirichlet problem is unique.

Let u_2 be the result of "sweeping out the Riesz measure" of u_1 out of the sector D. This means that

$$u_2(z) = \begin{cases} v(z) & \text{for } z \in D, \\ u_1(z) & \text{otherwise.} \end{cases}$$

Evidently, u_2 is subharmonic in the plane and satisfies (4.2). We shall prove that u_2 also satisfies the strict version of (1.6), namely

$$u_2(z) < u_2(|z|) \quad \text{for } z \notin [0, \infty). \tag{4.3}$$

In order to do so, we note first that u_1 is not harmonic in any neighborhood of the positive ray. This follows since $u_1(r)$ is not of the form $u_1(r) = c\log r$ and $u_1(re^{i\theta})$ does not depend on θ for $|\theta| \leq \varepsilon$. Because u_1 is subharmonic and v is harmonic this implies that $v(r) > u_1(r)$ for $r > 0$. As u_1 satisfies (1.6) we see that u_2 satisfies (4.3) for $\delta \leq |\arg z| \leq \pi$. In order to prove that u_2 satisfies (4.3) also for $|\arg z| \leq \delta$, let G be the plane cut along the negative ray and define

$$\psi_\alpha(z) = z^{\alpha/\pi} \quad \text{for } z \in G,$$

with the branch of the power chosen such that $\psi(z) > 0$ for $z > 0$. We claim that for $\alpha \in (\delta, \varepsilon)$, the function $v_\alpha = u_2 \circ \phi_\alpha$, extended by continuity to the negative ray, is subharmonic in the plane. Indeed, near the negative ray this function does not depend on $\arg z$ and it is subharmonic at all points except the negative ray, thus it is also subharmonic in a neighborhood of the negative ray.

The limit of these subharmonic functions v_α as $\alpha \to \delta + 0$ is the function v_δ which is thus subharmonic. But the Riesz measure of this function v_δ is supported on the negative ray, thus

$$v_\delta(z) = \int_0^1 \log|z+t|d\nu(t) + \int_{1+}^\infty \log\left|1 + \frac{z}{t}\right| d\nu(t),$$

with some non-negative measure ν. It is evident from this expression that for every $r > 0$ the function $t \mapsto v_\delta(re^{it})$ is strictly decreasing on $[0, \pi]$.

Thus for every $r > 0$, our function $t \mapsto u_2(re^{it})$ is strictly decreasing in the interval $[0, \delta]$. This, together with the fact that u_2 satisfies (4.2), completes the proof that u_2 satisfies (4.3).

3. Now we approximate our function u_2 by a function u_3 which is harmonic near 0. We set

$$u_3(z) = u_2(z + \varepsilon).$$

Then u_3 is harmonic near the origin, and using (4.3) and monotonicity of u_2 on the positive ray, we obtain

$$u_3(z) = u_2(z+\varepsilon) < u_2(|z+\varepsilon|) \leq u_2(|z| + \varepsilon) = u_3(|z|)$$

for $z \neq [0, \infty)$, so (4.3) is satisfied by u_3.

4. The subharmonic function u_3 we constructed has the following properties:

a) it satisfies (4.3),

b) it is harmonic near the origin,

c) it is harmonic in a neighborhood of the positive ray.

To construct a function which, in addition, is also harmonic near ∞ we consider the function

$$v(z) = u_3(1/z) + k \log |z|,$$

where $k = \mu_3(\mathbf{C})$, and μ_3 is the Riesz measure of u_3. It is easy to see that this function is subharmonic, if we extend it to 0 appropriately. Notice that v satisfies (4.3), and it is harmonic in an angular sector containing the positive ray (in fact in the sector $|\arg z| < \delta$). The function $w(z) = v(z + \varepsilon)$ also satisfies (4.3) by the same argument that we used in Step 3 to show that u_3 satisfies (4.3). Moreover, it is harmonic near the origin and near infinity. Thus the function

$$u_4(z) = w(1/z) + k \log |z|$$

has all properties a), b), c) and in addition

d) it is harmonic in a punctured neighborhood of infinity.

5. As u_4 is harmonic in a neighborhood of the origin, it has a representation

$$u_4(z) = u_4(0) + \int \log \left| 1 - \frac{z}{\zeta} \right| d\nu_4(\zeta).$$

Here ν_4 denotes the Riesz measure of u_4. As u_4 satisfies (4.2), we can write

$$u_4(x + iy) = u_4(0) + cx + O(z^2), \quad z = x + iy \to 0,$$

where

$$c = \frac{d}{dx} \left(\int \log \left| 1 - \frac{x}{\zeta} \right| d\nu_4(\zeta) \right) \bigg|_{x=0} = - \int \frac{\operatorname{Re} \zeta}{|\zeta|^2} d\nu_4(\zeta).$$

Property (4.3) of u_4 implies that $c \geq 0$. We may achieve $c > 0$ by adding to u_4 the potential $\varepsilon \log |1 + z|$. This procedure changes c to $c + \varepsilon$. This also makes positive the linear term in the expansion at ∞. Thus we obtain a function u_5, close to our original potential u in the weak topology, which besides (4.2) and (4.3) also satisfies

$$u_5(x + iy) = \nu_5(\mathbf{C}) \log |z| + b/x + O(z^{-2}), \quad z \to \infty, \tag{4.4}$$

$$u_5(x + iy) = u_5(0) + ax + O(z^2), \quad z = x + iy \to 0, \tag{4.5}$$

with positive constants a and b.

6. In our final step we replace the Riesz measure of u_5 by a nearby discrete probability measure with finitely many atoms, each having rational mass.

Let μ be the Riesz measure of u_5. If $\mu(\mathbf{C}) < 1$ we change μ to a probability measure by adding an atom sufficiently far at the negative ray. Evidently, this procedure does not destroy our conditions (4.2) and (4.3), and we also still have (4.4) and (4.5) for certain positive constants a and b.

By our construction, the support of μ is disjoint from the open set

$$H = \{z\colon |\arg z| < \delta\} \cup \{z\colon |z| < \delta\} \cup \{z\colon |z| > 1/\delta\},$$

and replacing δ by a smaller number if necessary we may assume that this also holds after the atom on the negative ray was added.

Let μ_k be any sequence of symmetric discrete measures each having finitely many atoms of rational mass, supported outside H, and $\mu_k \to \mu$ weakly. Let w_k be the potential of μ_k. Clearly the w_k satisfy (4.2). We show that they also satisfy (4.3), provided k is large.

First we consider small $|z|$, noting that the w_k are harmonic for $|z| < \delta$. For $z = re^{i\theta}$ with $0 < r < \delta$ we thus have the expansion

$$w_k(z) = \sum_{n=0}^{\infty} a_{n,k} r^n \cos n\theta. \tag{4.6}$$

Hence

$$\frac{\partial^2}{\partial\theta^2} w_k(z) = -a_{1,k} r \cos\theta + \Phi_k(z) \tag{4.7}$$

with

$$\Phi_k(z) = -\sum_{n=2}^{\infty} a_{n,k} r^n n^2 \cos n\theta.$$

As the w_k are harmonic for $|z| < \delta$, the convergence to u_5 is locally uniformly there, and $\partial^2 w_k/\partial\theta^2$ also converges there locally uniformly to $\partial^2 u_5/\partial\theta^2$. For $0 < \eta < b$ and large k we thus have $a_{1,k} > \eta$ by (4.5). Moreover, for $0 < r_0 < \delta$ there exists $C > 0$ such that $|w_k(z)| \le C$ for $|z| = r_0$ and all k. By Cauchy's inequalities we obtain $|a_{n,k} r_0^n| \le C_1$ and hence

$$|\Phi_k(z)| \le C_2 r^2 \quad \text{for } r \le r_0/2.$$

This inequality, together with (4.7) shows that w_k satisfies (4.3) for $|z| < r_1$ with some r_1 independent of k.

The case of large $|z|$ is treated similarly, using (4.4) and the transformation

$$u(z) \mapsto \log|z| + u(1/z), \tag{4.8}$$

as we did before. Thus there exists $r_2 > 0$ such that w_k satisfies (4.3) for $|z| > r_2$.

We finally consider the case that $r_1 \leq |z| \leq r_2$. Recall that by the first statement of Lemma 1, $\partial^2 u/\partial\theta^2$ is negative on the positive ray, so we have a positive constant c such that $(\partial^2/\partial\theta^2)u(re^{i\theta}) < -c$ in some angular sector

$$S := \{z\colon |\arg z| < \beta,\ r_1 \leq |z| \leq r_2\}.$$

We conclude that

$$L(r) := u(r) - u(re^{i\beta}) \geq c_1 > 0 \quad \text{for } r_1 \leq r \leq r_2.$$

On the interval $[r_1, r_2]$ the convergence $w_k \to u$ is uniform, because u and w_k are harmonic in S. On the other hand, on the compact set

$$K := \{z\colon r_1 \leq |z| \leq r_2,\ |\arg z| \geq \beta\}$$

we have $w_k(z) \leq u(z) + c_1/2$ for all sufficiently large k. This follows from the general convergence properties of potentials of weakly convergent measures summarized in the Appendix. We conclude that w_k satisfies (4.3) also for $r_1 \leq |z| \leq r_2$, and hence for all $z \in \mathbf{C}$.

Now w_k is the empirical measure of some polynomial

$$f(z) = a_0 + a_1 z + \ldots + a_{d-1} + a_d z^d,$$

and (4.3) implies that f satisfies (1.3). Clearly, $a_0 > 0$ and $a_d > 0$. Moreover, since $a_{1,k} > 0$ in (4.6), we see that $a_1 > 0$. The analogous expansion after the transformation (4.8) yields that $a_{d-1} > 0$. Thus the hypotheses of Theorem 1 are satisfied. Hence f^m has positive coefficients for some m. As the empirical measure of f and f^m coincide, we see that u_5 is a limit of empirical measures of polynomials with positive coefficients. As we may choose u_5 arbitrarily close to our original potential u by choosing ε sufficiently small, we see that u is also a limit of empirical measures of polynomials with positive coefficients. This completes the proof.

Appendix: Convergence of potentials

We frequently used various convergence properties of potentials of weakly convergent measures which we state here for the reader's convenience. An excellent references for all this material is [7].

Let $\mu_n \to \mu$ be a sequence of weakly convergent positive measures. This means that for every continuous function ϕ with bounded support

$$\int \phi d\mu_n \to \int \phi d\mu, \quad n \to \infty.$$

If we restrict here to C^∞-functions ϕ with bounded support, we obtain convergence in the space D' of Schwartz distributions. Actually, for positive measures weak convergence is equivalent to D'-convergence.

Now the sequence of subharmonic functions

$$u_n(z) = \int_{|\zeta|\leq 1} \log|z-\zeta| d\mu_n(\zeta) + \int_{|\zeta|>1} \log\left|1-\frac{z}{\zeta}\right| d\mu_n(\zeta)$$

converges in D' to the potential of the limit measure μ.

We cite Theorem 3.2.13 from [7] which says that this convergence of potentials also holds in several other senses.

Theorem A. *Let $u_j \not\equiv -\infty$ be a sequence of subharmonic functions converging in D' to the subharmonic function u. Then the sequence is uniformly bounded from above on any compact set. For every z we have*

$$\limsup_{n\to\infty} u_n(z) \leq u(z). \tag{4.9}$$

More generally, if K is a compact set, and $f \in C(K)$, then

$$\limsup_{n\to\infty} \sup_K (u_n - f) \leq \sup_K (u-f).$$

If $d\sigma$ is a positive measure with compact support such that the potential of $d\sigma$ is continuous, then there is equality in (4.9) *and $u(z) > -\infty$ for almost every z with respect to $d\sigma$. Moreover, $u_j d\sigma \to u d\sigma$ weakly.*

In this paper we deal with subharmonic functions satisfying (4.1), so $u(r)$ is increasing and convex with respect to $\log r$ on $(0,\infty)$. Choosing the length element on $[0,R]$ as $d\sigma$ in Theorem A, we conclude that $u_n \to u$ almost everywhere on the positive ray. For convex functions with respect to the logarithm this is equivalent to the uniform convergence on compact subsets of $(0, \infty)$. In particular, $u_n(r) \to u(r)$ at every point $r > 0$. As the u_n satisfy (1.6), we conclude that

$$\limsup_{n\to\infty} u_n(re^{i\theta}) \leq u(r).$$

Choosing the uniform measure on the circle $|z| = r$ as $d\sigma$ in Theorem A, we conclude that $u(re^{i\theta}) \leq u(r)$ almost everywhere with respect to $d\sigma$. As u is upper semi-continuous, we conclude that $u(re^{i\theta}) \leq u(r)$. Thus (1.6) is preserved in the limit.

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