More examples with Bessel functions

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1. Dirichlet problem for a cylinder

This problem describes *time-independent* solutions of the wave and heat equations in a cylinder.

A cylinder is described in cylindrical coordinates by inequalities

$$0 \leq r \leq L, \quad 0 \leq z \leq H,$$

where L is the radius and H is the height. We want to solve the Laplace equation in cylindrical coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0, \tag{1}$$

under the boundary conditions

$$u(r, \theta, 0) = f(r, \theta), \tag{2}$$

$$u(r, \theta, H) = g(r, \theta), \tag{3}$$

$$u(L, \theta, z) = h(\theta, z). \tag{4}$$

where f, g, h are some given functions. Besides, we have the periodicity conditions

$$u(r, -\pi, z) = u(r, \pi, z), \quad u_\theta(r, -\pi, z) = u_\theta(r, \pi, z), \tag{5}$$

which come from the nature of cylindrical coordinates.

As always, the problem is split into 3 problems:

a) $f = g = 0$,

b) $g = h = 0$,

c) $f = h = 0$,

and the complete solution is the sum of three solutions of a),b),c).

The first step is common for all three problems: separation of the variables. We look for solutions of the form $u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$. Plugging this form, we obtain

$$R''\Theta Z + \frac{1}{r}R'\Theta Z + \frac{1}{r^2}R\Theta'' Z + R\Theta Z'' = 0.$$

The general rule is to *separate those variables first for which the boundary conditions are homogeneous.* So for all problems a), b), c) we separate θ part first, since the conditions (5) are homogeneous. So we rewrite our equation as

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\frac{Z''}{Z} = -\frac{\Theta''}{\Theta}.$$

Both sides must be constant, say equal to μ, and for θ we obtain

$$\Theta'' + \mu^2\Theta = 0$$

with periodic boundary conditions. As we know, this implies that $\mu^2 = m^2$ where m is a non-negative integer, and to each positive m correspond two eigenfunctions, which can we written in the real form

$$\Theta_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta), \quad m = 0, 1, 2, \ldots,$$

or in the complex form

$$\Theta_m = c_m e^{im\theta}, \quad -\infty < m < +\infty.$$

(To $m = 0$ corresponds only one eigenfunction, a constant).

The remaining equation is

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\frac{Z''}{Z} - m^2 = 0,$$

and we separate r from z:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{m^2}{r^2} = -\frac{Z''}{Z}. \qquad (6)$$

In problem a),

the boundary conditions for z are homogeneous, (they come from (2), (3) with $f = g = 0$), so we consider the z part first. We have

$$Z'' + \lambda Z = 0, \quad Z(0) = Z(H) = 0.$$

This is a familiar problem: the eigenvalues ares

$$\lambda_n = \pi^2 n^2 / H^2, \tag{7}$$

and eigenfunctions are

$$Z_n(z) = \sin\left(\frac{\pi n z}{H}\right), \quad n = 1, 2, 3, \ldots.$$

The remaining equation in r then becomes

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{m^2}{r^2} - \lambda_n = 0,$$

or

$$r^2 \frac{R''}{R} + r\frac{R'}{R} - \left(\lambda_n r^2 + m^2\right) R = 0.$$

This looks like the equation reducible to the Bessel equation, except the wrong sign of λ_n (compare with eigenvalue problem for a disk). So it is reduced to Bessel by the change of the variable

$$R(r) = y\left(\sqrt{-\lambda_n}\, r\right) = y\left(i\frac{\pi n r}{H}\right), \quad i = \sqrt{-1}, \tag{8}$$

where y satisfies the Bessel equation

$$x^2 y'' + x y' + (x^2 - m^2) y = 0. \tag{9}$$

The general solution of this last equation is

$$y(x) = a J_m(x) + b Y_m(x),$$

where J_m and Y_m are Bessel functions of the first and second kind, and since our function R is supposed to be bounded, while $Y_m(x) \to -\infty$ as $x \to 0$, we must take $b = 0$.

So we obtain:

$$R_{m,n}(r) = J_m\left(i\frac{\pi n r}{H}\right).$$

Combining $R_{m,n}, \Theta_n, Z_m$, we obtain a series for problem a):

$$u(r,\theta,z) = \sum_{m,n} c_{m,n} J_m\left(i\frac{\pi n r}{H}\right) e^{im\theta} \sin\frac{\pi n z}{H}.$$

To satisfy the boundary condition (4) we plug $r = L$, and obtain a double Fourier series

$$h(\theta,z) = \sum_{m,n} c_{m,n} J_m(i\pi n L/H) e^{im\theta} \sin(\pi n z/H).$$

and Fourier formulas give us

$$c_{m,n} = \frac{1}{\pi H J_m(i\pi n L/H)} \int_{-\pi}^{\pi}\int_0^H h(\theta,z) e^{-im\theta} \sin(\pi n z/H) dz d\theta,$$

where we used the normalization factors

$$\int_{-\pi}^{\pi} |e^{im\theta}|^2 d\theta = 2\pi, \quad \text{and} \quad \int_0^H \sin^2(\pi n z/H) dz = \frac{H}{2}.$$

This solves problem a).

Remark. Functions $i^{-\nu} J_\nu(ix)$ are sometimes called *modified Bessel functions* and the standard notation for them is I_ν. The power of i multiple is added to make them real on the positive ray.

Problems b), and c).

We return to (6). This time the problem in r is homogeneous, so we consider the r-part first. Denoting the common value of the RHS and LHS of (6) by $-\lambda_m^2$ we obtain the r-part:

$$r^2 R'' + rR' + (\lambda_m^2 r^2 - m^2)R = 0, \tag{10}$$

with the boundary condition

$$R(L) = 0, \tag{11}$$

which comes from (4) with $h = 0$. This is reduced to Bessel equation by setting $R = y(\lambda_m r)$, so that y will satisfy (9), and taking into account that

$R(0)$ must be finite, we obtain $R = J_m(\lambda_m r)$. Now the boundary condition (11) implies

$$\lambda_{m,k} = x_{m,k}/L,$$

where $x_{m,k}$ is the k-th zero of J_m. Notice that $\lambda_{m,k}$ are all real since Bessel functions of order > -1 have only real roots, and without loss of generality we may consider only positive zeros since only λ_m^2 enters our equation (10).

Once $\lambda_{m,k}$ is found, it remains to solve the z part in (6). We have

$$Z'' - \lambda_{m,k}^2 Z = 0,$$

whose general solution can be written as

$$Z(z) = a\cosh(\lambda_{m,k}z) + b\sinh(\lambda_{m,k}z).$$

So the general solution satisfying homogeneous boundary conditions for the cases b), c) has the form

$$u_{m,k}(r,\theta,z) = \sum_{m,k} J_m(\lambda_{m,k}r)e^{im\theta}\left(a_{m,k}\cosh(\lambda_{m,k}z) + b_{m,k}\sinh(\lambda_{m,k}z)\right).$$

To satisfy the boundary conditions (3), (4), we plug $z = 0$ or $z = H$ and use Fourier-Bessel formulas.

For example, for problem c), the boundary condition at $z = 0$ is zero, so we set $a_{m,k} = 0$, and obtain

$$g(r,\theta) = \sum_{m,k} b_{m,k}J_m(x_{m,k}r/L)e^{im\theta}\sinh\left(x_{m,k}H/L\right),$$

and Fourier formulas give

$$b_{m,k} = \frac{1}{\pi L^2 J_{m+1}(x_{m,k})^2\sinh(x_{m,k}H/L)}\int_{-\pi}^{\pi}\int_0^L g(r,\theta)J_m(x_{m,k}r/L)e^{-im\theta}r\,dr\,d\theta.$$

Notice that we integrate rdr because Bessel functions are orthogonal with weight r, and we used the formula for the square norm

$$\int_0^L J_m^2(x_{m,k}r/L)r\,dr = \frac{L^2}{2}J_{m+1}(x_{m,k}),$$

which is formula (29) in the handout "Bessel functions".

2. Eigenvalue problem for Laplace equation in a cylinder

To solve heat of wave equation in a cylinder, we need the following eigenvalue problem

$$\Delta u + \lambda^2 u = 0,$$

with some boundary conditions. (I denoted the eigenvalue by λ^2 for convenience of some further formulas). Let us take for example,

$$u(x) = 0 \quad \text{on the boundary of the cylinder.}$$

In cylindrical coordinates this becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} + \lambda^2 u = 0,$$

$$u(L,\theta,z) = 0, \quad -\pi \le \theta \le \pi, \quad 0 \le z \le H,$$

$$u(r,\theta,0) = u(r,\theta,H) = 0, \quad 0 \le r \le L, \quad -\pi \le \theta \le \pi.$$

plus the periodicity condition (5) which comes from the nature of cylindrical coordinates.

As always, the first step is separation of variables, and plugging $u = R\Theta Z$ we obtain

$$R''\Theta Z + \frac{1}{r}R'\Theta Z + \frac{1}{r^2}R\Theta'' Z + R\Theta Z'' + \lambda^2 R\Theta Z = 0.$$

Separate θ first:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\frac{Z''}{Z} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta}. \tag{12}$$

We conclude that the separation constant must be of the form m^2, where m is a non-negative integer, and eigenfunctions are

$$\Theta_{\pm m}(\theta) = e^{\pm im\theta}, \quad m = 0, 1, 2, \ldots.$$

Replacing Θ''/Θ in (12) by $-m^2$ and dividing on r^2 we obtain

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{m^2}{r^2} + \lambda^2 = -\frac{Z''}{Z},$$

so the common value of the RHS and LHS is a constant, and we obtain a familiar problem for z variable, with boundary conditions $Z(0) = Z(H) = 0$, from which we conclude that this separation constant must be $\pi^2 n^2/H^2$ and

$$Z_n(z) = \sin\frac{\pi n z}{H}, \quad n = 1, 2, 3, \ldots.$$

Now for the r part we obtain

$$r^2 R'' + rR' + \left(\left(\lambda^2 - \frac{\pi^2 n^2}{H^2} \right) r^2 - m^2 \right) R = 0, \tag{13}$$

and this is reduced to the Bessel equation by the change of the variable

$$R(r) = f(\sqrt{\lambda^2 - \pi^2 n^2 / H^2}\, r).$$

Then as always, we conclude that the solution must be a scaled Bessel function J_m since the second linear independent solution of the Bessel equation is not bounded near 0. Moreover, we know that all roots of the Bessel function are real, so we must have

$$\lambda^2 \geq \pi^2 n^2 / H^2. \tag{14}$$

If equality holds in (14), then (13) is an Euler equation whose linearly independent solutions are $R(r) = r^m$ and $R(r) = r^{-m}$, when $m = 0$ the second solution is $\log r$. In any case, the second solution is infinite at 0, so it must be rejected. But then the solution $R(r) = r^m$ cannot satisfy the boundary condition $R(L) = 0$. Thus we must have a *strict* inequality in (14).

Thus

$$R(r) = J_m(\sqrt{\lambda^2 - \pi^2 n^2 / H^2}\, r),$$

and the boundary condition $R(L) = 0$ implies

$$\lambda^2_{m,n,k} = (x_{m,k}/L)^2 + (\pi n / H)^2. \tag{15}$$

So we found eigenvalues, and eigenfunctions are

$$u_{m,n,k}(r, \theta, z) = e^{im\theta} \sin \frac{\pi i z}{H} J_m(\lambda_{m,n,k} r).$$

Here m is any integer (positive or negative), while n and k are positive integers.

Exercise. Radial eigenvalue problem for Laplace equation in a ball.

Consider the eigenvalue problem

$$\Delta u + \lambda^2 u = 0$$

in a ball $|x| \leq L$ in 3-space, with zero boundary condition. Here one has to use spherical coordinate's (r, ϕ, θ) which will be fully discussed later. But let us restrict the problem to eigenfunctions which depend on the radius $r = |x|$ only. Find all such eigenfunctions and corresponding eigenvalues.

For radial functions in spherical coordinates, the expression of the Laplacian in the following

$$\Delta_r u = u_{rr} + \frac{2}{r} u_r.$$

Here subscript r in Δ_r means that we drop all terms which involve differentiation with respect to ϕ, θ.

So we want to solve the equation

$$y'' + \frac{2}{r} y' + \lambda^2 y = 0, \tag{16}$$

with boundary conditions that $y(0)$ is finite, and

$$y(L) = 0.$$

I suggest two ways of solving this problem.

1. Try to find a simple change of the variable which reduces this equation to a Bessel equation.

2. If you cannot do this, try to solve it using power series. The associated Euler equation is

$$r^2 y'' + 2ry' = 0,$$

where we dropped the term $\lambda^2 r^2$. The corresponding characteristic equation is

$$\rho^2 + \rho = 0,$$

so solutions of Euler's equation are 1 and $1/r$. This suggests that a solution of (16) which is bounded at 0 must be of the form

$$y(r) = \sum_{n=0}^{\infty} a_n r^n.$$

Plug this form, determine the coefficients a_n explicitly, and recognize the resulting function y, it turns out to be elementary. This permits you to find eigenvalues and eigenfunctions.

Once you have an explicit solution of (16), you can return to step 1, and find a simple change of the variable which reduces it to Bessel equation.