Bessel functions

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1. Eigenvalue problem for the Laplacian in a disk.

Consider the eigenvalue problem for the Laplace operator[1]

$$\Delta u + \lambda^2 u = 0$$

with zero boundary conditions in the disk described in polar coordinates by the inequality $x^2 + y^2 < L^2$. So the boundary conditions are

$$u(L, \theta) = 0, \quad 0 \leq \theta \leq 2\pi,$$

and the function $\theta \mapsto u(r, \theta)$ is smooth, 2π-periodic.

Expressing the Laplacian in polar coordinates we obtain

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \lambda^2 u = 0.$$

Looking for solutions of the form $u(r, \theta) = R(r)\Theta(\theta)$, we obtain

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' + \lambda^2 R\Theta = 0.$$

We divide on $R\Theta$, multiply on r^2 and move the term with θ to the RHS:

$$r^2 \frac{R''}{R} + r \frac{R}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta}.$$

Since the LHS depends only on r and the LHS only on θ, they are both eual to the same constant. Then for θ-part we obtain the familiar eigenvalue

[1] I denoted the eigenvalue by λ^2 because I know in advance that it is going to be positive, this convention will simplify some formulas.

problem, with periodic boundary conditions, and conclude that this constant (the common value of the LHS and RHS of (4)) must be a square of an integer, say n^2, and the eigenfunctions of the θ-problem are

$$\cos(n\theta), \quad n = 0, 1, \dots \quad \text{and} \quad \sin(n\theta), \quad n = 1, 2, \dots .$$

The r-part now becomes

$$r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0. \tag{1}$$

We can easily remove the dependence on λ by the change of the variables

$$R(r) = f(\lambda r), \quad x = \lambda r. \tag{2}$$

Our equation becomes

$$x^2 f'' + xf' + (x^2 - \nu^2)f = 0, \quad \text{where} \quad \nu = n, \tag{3}$$

which is the standard form of the **Bessel equation**. The reason we introduced a new letter ν is that we are going to study it not only for integer values of ν, for further applications. This number ν is called the **order** of the Bessel equation.

This equation cannot be solved in elementary functions (unless $\nu = 1/2+$an integer), so we have to study its solutions by themselves, using the equation.

2. Bessel functions

Equation (3) differs from Euler's equation by the term x^2 in parentheses. This suggests that we may look for a solution of the form

$$f(x) = x^b \sum_0^\infty a_j x^j = \sum_0^\infty a_j x^{j+b}, \quad a_0 \neq 0. \tag{4}$$

where x^b is a solution of the Euler equation.

By differentiating this expression for f twice, we obtain

$$xf'(x) = \sum_0^\infty (j+b) a_j x^{j+b}, \tag{5}$$

$$x^2 f''(x) = \sum_0^\infty (j+b-1)(j+b) a_j x^{j+b}, \tag{6}$$

and we transform the term $x^2 f(x)$ as

$$x^2 f(x) = \sum_{k=0}^{\infty} a_k x^{k+b+2} = \sum_{j=2}^{\infty} a_{j-2} x^{j+b}, \tag{7}$$

where we changed the summation index k to $j = k+2$. Plugging (4), (5), (6), (7) into (3) we must obtain an identity, that is the coefficient at x^{j+b} for each $j \geq 0$ must be zero.

This gives:

$$\begin{aligned}
&\text{for} \quad j = 0: \qquad && (b^2 - \nu^2) a_0 = 0, && (8)\\
&\text{for} \quad j = 1: \qquad && ((b+1)^2 - \nu^2) a_1 = 0, && (9)\\
&\text{for} \quad j \geq 2: \qquad && ((b+j)^2 - \nu^2) a_j + a_{j-2} = 0.. && (10)
\end{aligned}$$

Since we assume $a_0 \neq 0$ we obtain from (8) $b = \pm\nu$, as expected from comparison with Euler's equation (equation $b^2 - \nu^2 = 0$ is the characteristic equation of this Euler equation). One can always satisfy (9) by setting $a_1 = 0$, which we do. Then (10) gives us a two-step recurrence relation:

$$a_j = -\frac{a_{j-2}}{(j+b)^2 - \nu^2} = -\frac{a_{j-2}}{j(j+2b)}, \quad j = 2, 3, \ldots, \tag{11}$$

where we used $b^2 = \nu^2$.

When ν is real (we are only concern with this case), we may suppose without loss of generality that $\nu \geq 0$, since only ν^2 enters our equation (3). Then, there can be a problem with (11) if b is a negative integer, the case we leave aside for a while. If $b = \nu \geq 0$, recurrent relation can be solved, and we obtain

$$a_2 = -\frac{a_0}{2(2+2\nu)}, \quad a_4 = \frac{a_0}{2 \cdot 4(2+2\nu)(4+2\nu)},$$

and so on.

Then,

$$\begin{aligned}
a_{2k} &= \frac{(-1)^k a_0}{2 \cdot 4 \cdot \ldots \cdot (2k)(2+2\nu)(4+2\nu)\ldots(2k+2\nu)}\\
&= \frac{(-1)^k a_0}{2^{2k} k! (1+\nu)(2+\nu)\ldots(k+\nu)},
\end{aligned}$$

and

$$a_{2k+1} = 0, \quad k = 0, 1, 2, \ldots.$$

At this point, it is convenient to use Gamma function (see the handout "Some useful integrals"). We choose

$$a_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

and use the formula

$$\Gamma(k+1+\nu) = (k+\nu)(k+\nu-1)\dots(1+\nu)\Gamma(1+\nu).$$

This permits to write the solution that we obtained as

$$f(x) = \sum_0^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} = x^\nu \sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{2^{2k+\nu}k!\Gamma(k+\nu+1)} =: J_\nu(x). \tag{12}$$

This function is called the **Bessel function** (of the first kind) of order ν. One can easily show that the radius of convergence of the power series at the end of (12) is infinite, so the power series converges for all complex x.

When $x \to 0$, and ν is not an integer, we have

$$J_\nu(x) \sim x^\nu \quad x \to 0, \tag{13}$$

so J_ν and $J_{-\nu}$ are linearly independent. We have $J_0(0) = 1$, and $J_\nu(0) = 0$ when $\nu > 0$. Since for non-integer ν, J_ν and $J_{-\nu}$ satisfy the same differential equation (3), and are linearly independent. we conclude that the general solution of this equation is

$$aJ_\nu(x) + bJ_{-\nu}(x).$$

Now let us suppose now that $\nu = n$ is a positive integer, and try to define J_{-n} by the same formula (12). I recall that Γ equals ∞ at negative integers, so since it stands in the denominators of the formula (12) all terms with $k < n$ vanish. For $k \geq n$ we have $\Gamma(k-n+1) = (k-n)!$, so changing the summation index to $j = k-n$ we obtain:

$$J_{-n} = \sum_{k=n}^\infty \frac{(-1)^k}{k!(k-n)!}\left(\frac{x}{2}\right)^{2k-n} = \sum_{j=0}^\infty \frac{(-1)^{j+n}}{j!(j+n)!}\left(\frac{x}{2}\right)^{2j+n} = (-1)^n J_n(x).$$

So $J_n = (-1)^n J_{-n}$ and for integer ν we obtained only one linearly independent solution of the Bessel equation.

The second one is usually taken to be the **Bessel function of the second kind** which is defined first for non-integer ν by the formula

$$Y_\nu(x) := \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}, \tag{14}$$

and then for integer n as the limit

$$Y_n(x) := \lim_{\nu\to n} Y_\nu(x).$$

One can show that the limit exists (it is an indeterminate expression of the form $0/0$ when ν is integer in (14)), and has these properties:

$$Y_n(x) \sim -\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n}, \quad x \to 0, \quad n \geq 1, \tag{15}$$

and

$$Y_n(x) \sim \frac{2}{\pi}\log\frac{x}{2}, \quad x \to 0. \tag{16}$$

For this course, it will be only important that $Y_n(x) \to -\infty$ as $x \to 0$.

To conclude this part, we state that the general solution of the Bessel equation of integer order $\nu = n$ is

$$aJ_n(x) + bY_n(x).$$

3. Properties of Bessel functions.

The following properties are obtained by simple manipulations with the series:

$$\begin{aligned}
\left(x^{-\nu}J_\nu(x)\right)' &= -x^{-\nu}J_{\nu+1}(x), && (17)\\
(x^\nu J_\nu(x))' &= x^\nu J_{\nu-1}(x), && (18)\\
xJ_\nu'(x) - \nu J_\nu(x) &= -xJ_{\nu+1}(x), && (19)\\
xJ_\nu'(x) + \nu J_\nu(x) &= xJ_{\nu-1}(x), && (20)\\
xJ_{\nu-1}(x) + xJ_{\nu+1}(x) &= 2\nu J_\nu(x), && (21)\\
J_{\nu-1}(x) - J_{\nu+1}(x) &= 2J_\nu'(x). && (22)
\end{aligned}$$

(See p. 133 of the book for proofs).

As it was already mentioned, Bessel's functions of half-integer order are elementary. We begin with $J_{-1/2}$, use the properties of Γ and slightly rearrange the terms (see p. 134 of the book):

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_0^\infty \frac{(-1)^k x^{2k}}{(2k)!} = \sqrt{\frac{2}{\pi x}} \cos x.$$

Similarly,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

Then formula (21) shows that $J_{k+1/2}$ is an elementary function for all integers k.

Exercise. Compute $J_{3/2}$ and $J_{5/2}$.

Even more important property is the following formula:

$$\sum_{-\infty}^{\infty} J_n(x) z^n = \exp\left(\frac{x}{2}\left(z - \frac{1}{z}\right)\right). \tag{23}$$

This is also obtained by direct manipulation with power series (see p. 134-135 of the book). Plugging $z = e^{it}$ we obtain from (23)

$$e^{ix\sin t} = \sum_{-\infty}^{\infty} J_n(x) e^{int},$$

so we obtained the Fourier expansion of the function $t \mapsto \exp(ix \sin t)$, where x is a parameter! Applying Fourier's formulas to this expansion, we obtain

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\sin(t) - int} dt.$$

Since the LHS is real for real x, we can take the real part of the RHS:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x\sin(t) - nt) dt = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin(t) - nt) dt,$$

where we used the fact that cos is even and sin is odd.

This gives a new representation of the Bessel function of integer order, which can be taken as an alternative definition.

Exercise. Derive from (23) the identity

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y).$$

4. Zeros and asymptotics.

Power series gives a very good approximation of $J_\nu(x)$ for small x, but it is not useful when x is large.

Exercise. How many terms of the Taylor series you need to use to compute e^{-10} with error 1 percent?

A very good approximation for large x is given by the formula

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + E_\nu(x), \quad x > 1, \tag{24}$$

where $|E_\nu(x)| \leq c_\nu x^{-3/4}$, $x > 1$, and c_ν is some constant depending on ν. When $\nu = \pm 1/2$, this formula gives an exact equality, that is $c_{\pm 1/2} = 0$.

This approximation is obtained by substituting $f(x) = x^{-1/2} g(x)$ to the Bessel equation, which gives

$$g''(x) + g(x) + \frac{1/4 - \nu^2}{x^2} g(x) = 0.$$

In this equation, the last summand is small when x is large, so it is reasonable to expect that solutions will be close to solutions of $g'' + g = 0$, and the general solution of this last equation is $a\cos(x+b)$. It remains to determine constants a and b, to approximate the specific solution J_ν of the original equation.

Notice that this approximation (24) is actually very good, even when x is not too large, as you can see from the graphs (Handout: Plots of some Bessel functions). So for all practical purposes, the power series (3) together with approximation (24) are sufficient.

Approximation (24) permits also to approximate the positive zeros of Bessel functions, that is solutions of $J_\mu(x) = 0$. All except possibly one of them are close to the zeros of $\cos(z - \pi\nu/2 - \pi/4)$. So we have a sequence of zeros $x_1 < x_2 < \ldots \to +\infty$ on the positive ray, which are given by the approximate formula

$$x_k \approx \pi(k + m(\nu) + \nu/2 + 1/4),$$

where $m(\nu)$ is an integer depending on ν.

Exercise. Guess what this integer $m(\nu)$ is, exactly, by inspecting the graphs in the handount "Plots of Bessel functions".

Here is a little table of smallest positive zeros of J_n for $0 \le n \le 4$:

$n = 0$: 2.404825558, 5.520078110, 8.653727913, 11.79153444, 14.93091771,
$n = 1$: 3.831705970, 7.015586670, 10.17346814, 13.32369194, 16.47063005,
$n = 2$: 5.135622302, 8.417244140, 11.61984117, 14.79595178, 17.95981949,
$n = 3$: 6.380161896, 9.761023130, 13.01520072, 16.22346616, 19.40941523,
$n = 4$: 7.588342435, 11.06470949, 14.37253667, 17.61596605, 20.82693296,
$n = 5$: 8.771483816, 12.33860420, 15.70017408, 18.98013388, 22.21779990,

An important properties of zeros is that they are *interlacent*: between any two positive zeros of J_n there is exactly one zero of J_{n+1}. But on the first interval $(0, x_{n,1})$ there is no zero of J_{n+1}.

5. A singular Sturm-Liouville problem.

Recall section 1, where we arrived at the problem of solving the equation

$$r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0. \tag{25}$$

with the boundary condition $R(L) = 0$. In general, one needs two boundary conditions for a second order equation; the second one will be discussed shortly.

Dividing on r, we can rewrite the equation in the Sturm-Liouville form:

$$(rR')' - \frac{n^2}{r}R + \lambda^2 rR = 0, \tag{26}$$

so we have a formally self-adjoint differential operator

$$L(R) = (rR')' - \frac{n^2}{r}R$$

and the weight function $w(r) = r$. The problem is singular: the coefficient at R'' is equal to zero at $r = 0$, and the function n^2/r equals ∞ at 0. So the theory of regular Sturm-Liouville problems does not apply.

Now we know how to obtain general solution of (25): by changing $R(r) = f(\lambda r)$ we obtain Bessel's equation (3) for f, so the general solution of (25) is

$$aJ_n(\lambda r) + bY_n(\lambda r). \tag{27}$$

Now, recalling what our solutions describe (section 1) we conclude that the value $R(0)$ must be finite, and *this plays the role of the second boundary condition*, since Y_μ tends to $-\infty$ as $x \to 0$, while $J_n(0)$ is finite when $n \geq 0$, we conclude that $b = 0$ in (27). Then the boundary condition on the right end reads:

$$J_n(\lambda L) = 0, \tag{28}$$

and we conclude that λL must be one of the zeros of J_n.

It turns out that *for $\nu > -1$ all solutions of $J_\nu(x) = 0$ are real, and all except possibly $x = 0$ are simple.*

Since $x^{-\nu} J_\nu$ is an even function (see (3), solutions of (28) are summetric with respect to the origin. So the only positive solutions are

$$\lambda_{\nu,k} = x_{\nu,k}/L$$

where $x_{\nu,k}$ is the sequence of positive zeros of J_ν described in the previous section.

So we obtain a sequence of eigenvalues $\lambda_{n,k}^2$ and a sequence of eigenfunctions $R(r) = \phi_{n,k}(r) = J_n(\lambda_{n,k} r)$ of the problem (25) with boundary conditions: $R(0+)$ is finite and $R(L) = 0$.

It turns out that this system of eigenfunctions is a *complete orthogonal system with weight $w(r) = r$*:

Theorem. *Suppose that $\nu \geq 0$, $L > 0$ and $w(r) = r$, and Let $x_{\nu,k}$ we the sequence of positive zeros of the Bessel function J_ν. Then functions $\phi_{\nu,k}(r) = J_\nu(x_{\nu,k} r/L)$, $k = 1, 2, \ldots,$ form a complete w-orthogonal systems with*

$$\|\phi_{\nu,k}\|_w^2 = \frac{L^2}{2} J_{\nu+1}^2(x_{\nu,k}). \tag{29}$$

This means that every function $g \in L_w^2(0, L)$ can be expanded into a *Fourier–Bessel series*:

$$f(r) = \sum_{k=1}^{\infty} c_{\nu,k} \phi_{\nu,k},$$

$$c_{\nu,k} = \frac{1}{\|\phi_{\nu,k}\|^2} \int_0^L f(r)\phi_{\nu,k}(r) r dr.$$

As always, we skip the proof of completeness. But orthogonality and formula (29) are not hard to check (Lemma 5.4 on p. 147 of the book).

6. Completion of the investigation of the eigenvalue problem for the disk.

Now we can finish the study of the eigenvalue problem for the Laplacian in the disk which began in section 1. We know now that equation (1) has non-zero solutions satisfying boundary conditions (that $R(0+)$ is finite and $R(L) = 0$) if and only if $\lambda = \lambda_{n,k} = x_{n,k}/L$ where $x_{n,k}$ is the k-th positive zero of J_n. So eigenvalues are

$$\lambda_{n,k}^2 = x_{n,k}^2/L^2, \tag{30}$$

and eigenfunctions are

$$J_n(\lambda_{n,k} r)(a\cos(n\theta) + b\sin(n\theta)). \tag{31}$$

So, for example, if we want to decribe the sound of a circular membrane (a tamburin) of radius L, this amounts to solving the wave equation

$$u_{tt} = c^2 \Delta u$$

with boundary conditions $u(L, \theta, t) = 0$. Separating time from space variables we obtain $u(r, \theta, t) = T(t)v(r, \theta)$,

$$\frac{T''}{c^2 T} = \frac{\Delta v}{v} = -\lambda^2.$$

The space part that we just solved tells us that and the solution is a linear combination of oscillations

$$\begin{aligned} u(r, \theta, t) \quad &= \quad \sum_{n,k} \cos(c\lambda_{n,k} t) J_n(\lambda_{n,k} r)(a_{n,k} \cos(n\theta) + b_{n,k} \sin(n\theta) && (32) \\ &+ \quad \sin(c\lambda_{n,k} t) J_n(\lambda_{n,k} r)(a'_{n,k} \cos(n\theta) + b'_{n,k} \sin(n\theta) && (33) \end{aligned}$$

Here $c\lambda_{n,k} = cx_{n,k}/L$ are frequencies, and (31) are modes corresponding to each frequency.

To solve the initial value problem, say with

$$u(r, \theta, 0) = f(r, \theta) \quad \text{and} \quad u_t(r, \theta, 0) = g(r, \theta)$$

we plug $t = 0$ and expand f into a double Fourier–Bessel series:

$$f(r, \theta) = \sum_{n,k} J_n(x_{n,k} r/L)(a_{n,k} \cos(n\theta) + b_{n,k} \sin(n\theta))$$

Fourier–Bessel formulas and (29) give us the coefficients:

$$a_{n,k} = \frac{2}{\pi L^2 J_{n+1}^2(x_{n,k})} \int_{-\pi}^{\pi} \int_0^L f(r, \theta) J_n(x_{n,k} r/L) \cos(n\theta) r dr d\theta,$$

and similarly for $b_{n,k}$.

To satisfy the second boundary condition $u_t(r, \theta, 0) = g(r, \theta)$, one differentiates in t, plugs $t = 0$ and used Fourier–Bessel formulas for g, to determine $a'_{n,k}$ and $b'_{n,k}$.

Example. *Find the smallest and second smallest frequencies of oscillation of a membrane with clamped edge, of radius 10 cm, if the speed of wave propagations in this membrane is c =200 m/sec.*

The frequency of oscillation with mode

$$J_n(x_{n,k} r/L) \cos(n\theta)$$

is

$$\frac{c\lambda_{n,k}}{2\pi} = \frac{c x_{n,k}}{2\pi L} = \frac{2000}{2\pi} x_{n,k}.$$

From the table at the end of section 4, the two smallest values of $x_{n,k}$ are $x_{0,1} \approx 2.404$ and $x_{1,1} \approx 3.83$. So the frequencies are $\approx 765 Hz$ and $1219 Hz$.