Bloch radius, normal families and quasiregular mappings

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Abstract

Bloch's Theorem is extended to K-quasiregular maps $\mathbf{R}^n \to \mathbf{S}^n$, where $\mathbf{S}^n$ is the standard n-dimensional sphere. An example shows that Bloch's constant actually depends on K for $n \geq 3$.

Let $B(a,r) := \{x \in \mathbf{R}^n : |x-a| < r\}$ be an open ball, $0 < r \leq \infty$. Consider an open discrete map $f : B(0,r) \to M$ where M is a Riemannian manifold of dimension n. For every $x \in B(0,r)$ we define $d_f(x)$ as the radius of the maximal open ball $B \subset M$ centered at $f(x)$, such that a continuous right inverse ϕ with the property $\phi(f(x)) = x$ exists in B. If there is no such ball $d_f(x) = 0$. The Bloch radius of f is defined as

$$\mathfrak{B}(f) := \sup_{x \in B(0,r)} d_f(x).$$

The cases of principal interest are $M = \mathbf{R}^n$ with the standard Euclidean metric $|dx|$, and $M = \mathbf{S}^n$, the sphere with the standard spherical metric $2|dx|/(1+|x|^2)$, so that the diameter of the sphere in this metric is π. We denote the corresponding Bloch radii by $\mathfrak{B}_e(f)$ and $\mathfrak{B}_s(f)$, respectively. Notice that $\mathfrak{B}_s(f) \leq 2\mathfrak{B}_e(f)$, if we consider the push forward of the spherical metric to $\mathbf{R}^n$ via the stereographic projection. Notation $\mathfrak{B}(f)$ will be used in statements which are true for both metrics.

A family $\mathcal{F}$ of continuous maps $B(0,r) \to M$ is called normal if every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compacta. In the case of compact M a family is normal if and only if its restriction to every compact in $B(0,r)$ is equicontinuous [1]. The classical theorem of Bloch [3] can be stated as:

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Theorem A *Every family of holomorphic maps* $\mathbf{C} \supset B(0,1) \to \mathbf{C}$ *with bounded Euclidean Bloch radius is equicontinuous with respect to the Euclidean metric.*

A related result belongs to Valiron [13]:

Theorem B *Every non-constant entire function has infinite Euclidean Bloch radius.*

The case of spherical metric was first considered by Minda [8]:

Theorem C (i) *There exists an absolute constant* $b_0 > 0$*, such that the family of all meromorphic functions* $\mathbf{C} \supset B(0,1) \to \overline{\mathbf{C}}$ *whose spherical Bloch radius is at most* $b_0 - \epsilon$*, is normal for every* $\epsilon \in (0, b_0)$.
(ii) *For every non-constant meromorphic function* $f : \mathbf{C} = B(0,\infty) \to \overline{\mathbf{C}}$ *we have* $\mathfrak{B}(f) \geq b_0$.

It is conjectured that $b_0 = \arccos(1/3) \approx 70°32'$. Some recent results about precise constants are contained in [5]. The proofs of Theorems A-C are usually based on specific properties of holomorphic functions, like Taylor series expansion ([3, 13]) or on Gauss curvature considerations [2, 8].

In this paper we show that all results mentioned above, as well as their n-dimensional generalizations follow from a simple normal families argument, which is due to Zalcman [14] in dimension 2 and to Ruth Miniowitz [9] in arbitrary dimension. The survey [15] describes other applications of this tool.

The natural framework in dimensions greater than 2 is the class of quasiregular maps (see [10, 11] for the general theory of these maps). We recall that a continuous map is called K-quasiregular if its generalized partial derivatives are locally summable in degree n and the derivative D_f satisfies

$$\|D_f(x)\|^n \leq K|\det D_f(x)| \quad \text{almost everywhere.}$$

Non-constant quasiregular maps are open and discrete, according to [10, II §6.3]). Apparently they were first recognized by Bochner [4], as early as in 1946, as the appropriate class for extension of Geometric Function Theory to higher dimensions, the point of view widely shared today. Bochner showed that Theorem A extends to K-quasiregular maps $\mathbf{R}^n \supset B(0,1) \to \mathbf{R}^n$, whose coordinates are *harmonic functions*. For further results in this direction see [7].

Theorem 1 (i) *There exists a constant* $b(n,K) > 0$*, such that the family of all* K*-quasiregular maps* $\mathbf{R}^n \supset B(0,1) \to \mathbf{S}^n$*, whose spherical Bloch radii*

are at most $b(n,K)-\epsilon$, *is normal for every* $\epsilon \in (0, b(n,k))$.
(ii) *For every non-constant* K*-quasiregular map* $f : \mathbf{R}^n \to \mathbf{S}^n$ *we have* $\mathfrak{B}_s(f) \geq b(n,K)$.
(iii) *Every family of* K*-quasiregular maps* $\mathbf{R}^n \supset B(0,1) \to \mathbf{R}^n$ *with bounded Euclidean Bloch constant is equicontinuous with respect to the Euclidean metric.*
(iv) *Every non-constant* K*-quasiregular map* $f : \mathbf{R}^n \to \mathbf{R}^n$ *satisfies* $\mathfrak{B}_e(f) = \infty$.

Remarks The proof of Theorem 1 is a pure existence proof. It is desirable to find some way to estimate $b(n,K)$ effectively for $n \geq 3$. For $n = 2$ the spherical Bloch constant $b_0 = b(2,K)$ is actually independent of K. This follows from the fact that every quasiregular map $f : \mathbf{C} \to \overline{\mathbf{C}}$ can be factored as $f = g \circ \phi$, where g is meromorphic in $\mathbf{C}$, and ϕ is a homeomorphism, and it is clear that $\mathfrak{B}(f) = \mathfrak{B}(g)$. The situation is different for $n \geq 3$ as the following example shows. It is similar to the situation with "Picard's constant" which is equal to 3 for quasiregular maps in dimension 2, but depends on K in higher dimensions, as the example of Rickman [12] shows. Smooth[1] quasiregular maps in dimension at least 3 are locally injective, and thus by a theorem of Zorich [11] smooth quasiregular maps $\mathbf{R}^n \to \mathbf{R}^n$ are bijective, so (ii) and (iv) are immediate for smooth quasiregular maps.

Example 1 *For every* $n \geq 3$ *and every* $\epsilon > 0$ *there exists a quasiregular map* $\mathbf{S}^n \to \mathbf{S}^n$ *of degree* 2, *such that* $\mathfrak{B}_s(f) < \epsilon$. *There exists also an infinitely differentiable open discrete map* $f : \mathbf{S}^n \to \mathbf{S}^n$ *with* $\mathfrak{B}_s(f) < \epsilon$.

Consider the standard embedding $\mathbf{S}^{n-2} \subset \mathbf{S}^n$, and introduce the cylindrical coordinates on the complement $D := \mathbf{S}^n \backslash \mathbf{S}^{n-2} = \{(r,\theta,y) : r > 0, \theta \in [0,2\pi), y \in \mathbf{S}^{n-2}\}$. The "winding map" $D \to D$ given in cylindrical coordinates by $(r,\theta,y) \mapsto (r, 2\pi\{\theta/\pi\}, y)$, where $\{.\}$ stands for the fractional part, extends by continuity to the 2-quasiregular map $g : \mathbf{S}^n \to \mathbf{S}^n$, which is locally homeomorphic in D, and locally 2-to-1 at every point of $\mathbf{S}^{n-2}$. If $B \subset \mathbf{S}^n$ is a ball then a right inverse to g in B exists if and only if $B \cap \mathbf{S}^{n-2} = \emptyset$. Now we postcompose g with a diffeomorphism $h : \mathbf{S}^n \to \mathbf{S}^n$ such that $h(\mathbf{S}^{n-2})$ forms an ϵ-net, that is $\mathrm{dist}_s(x, h(\mathbf{S}^{n-2})) < \epsilon$ for every $x \in \mathbf{S}^n$. It is clear that $\mathfrak{B}_s(h \circ g) < \epsilon$. To construct an infinitely smooth map with similar properties, we replace the winding map by $(r,\theta,y) \mapsto (r^2, 2\pi\{\theta/\pi\}, y)$. □

The normality argument mentioned above is the following

[1] C^3 for $n = 3$ and C^2 for $n \geq 4$, see [11, p. 12]

Lemma 1 ([9]) *Let $\mathcal{F}$ be a family of K-quasiregular maps $\mathbf{R}^n \supset B(0,1) \to \mathbf{S}^n$ which is not normal. Then there exist $r \in (0,1)$ and sequences $f_m \in \mathcal{F}$, $x_m \in B(0,r)$ and $\rho_m > 0$, $\rho_m \to 0$, such that $g_m(x) := f_m(x_m + \rho_m x) \to f(x) \neq \text{const}$ uniformly on compacta in $\mathbf{R}^n$, and $f : \mathbf{R}^n \to \mathbf{S}^n$ is K-quasiregular. Moreover, we have for $x_1, x_2 \in B(0,R)$*

$$\text{dist}(f(x_1), f(x_2)) \leq 2(1+R^2)^\alpha |x_1 - x_2|^\alpha, \quad \textit{where } \alpha = (K)^{1/(1-n)}$$

and

$$\text{diam} f(B(0,1)) \geq \delta > 0,$$

where diam *is the diameter with respect to the spherical metric, and δ is a constant depending only of K and n.*

Proof of Theorem 1. First we notice that $\mathfrak{B}_s(f) > 0$ for every open discrete map. Recall that $x \in B(0,1)$ is called *critical* if there is no neighborhood V of x such that $f|V$ is a homeomorphism onto its image. The set of all critical points is closed and its topological dimension is at most $n-2$ [6]. So there is a point $a \in B(0,1)$ such that the restriction of f onto some ball $B(a,r)$ is a homeomorphism onto the image, so $\mathfrak{B}_s(f) > 0$.

Second we notice the following semicontinuity property of the Bloch's radius: if $g_m \to f$ uniformly on compacta then

$$\mathfrak{B}(f) \leq \liminf \mathfrak{B}(g_m). \tag{1}$$

To prove this property we fix arbitrary $\epsilon \in (0, \mathfrak{B}(f)/4)$ and put $r := \mathfrak{B}(f) - 2\epsilon$. Then there exists a ball $B(a, r+\epsilon)$ in which a continuous right inverse ϕ to f is defined. Put $\overline{D} := \phi(\overline{B}(a,r))$, this is an imbedding of the closed ball. As $g_m \to f$ uniformly on $\overline{D}$, we conclude for large values of m that $g_m(\partial\overline{D})$ is contained in ϵ-neighborhood of $\partial B(a,r)$. So the degree $\mu(y, g_m, D)$ is defined for every $y \in B(a, r-\epsilon)$ [10, II, §2]. Since the degree $\mu(y, f, D)$ for $y \notin f(\partial D)$ is continuous with respect to f, we conclude that $\lim \mu(y, g_m, D) = \mu(y, f, D) = \pm 1$ for every $y \in B(a, r-\epsilon)$. This means that for large m the restrictions $g_m|D$ have continuous right inverses in $B(a, r-\epsilon)$ so $\mathfrak{B}(g_m) \geq r - \epsilon \geq \mathfrak{B}(f) - 3\epsilon$, which proves (1).

To prove (i) in Theorem 1 by contradiction, we assume that for every $\epsilon > 0$ the family $\mathcal{F}_\epsilon$ consisting of all K-quasiregular maps $f : \mathbf{R}^n \supset B(0,1) \to \mathbf{S}^n$ with $\mathfrak{B}_s(f) \leq \epsilon$ is not normal. Applying Lemma 1 to each $\mathcal{F}_\epsilon$ we obtain a family $\{f_\epsilon : \epsilon > 0\}$ of quasiregular maps $\mathbf{R}^n \to \mathbf{S}^n$. This family is normal and has no constant limit functions because of the uniform estimates in Lemma 1 and we have $\mathfrak{B}(f_\epsilon) \leq \epsilon$. So we can find a convergent sequence

$f_{\epsilon_k} \to f \neq$ const with $\mathfrak{B}(f_{\epsilon_k}) \to 0$. We have $\mathfrak{B}_s(f) > 0$ because f is open, and this contradicts (1).

To prove (ii) it is enough to notice that the family $\{x \mapsto f(2^n x) : n \in \mathbf{N}\}$ with a non-constant function f is never normal in $B(0,1)$.

To prove (iii) we fix arbitrary $M > 0$ and consider the family $\mathcal{F}_M$ consisting of all K-quasiregular maps $\mathbf{R}^n \supset B(0,1) \to \mathbf{R}^n$ with the property $\mathfrak{B}_e(f) \leq M$. Put $\kappa = b(n,K)/(3M)$, where $b(n,K)$ is the constant from (i). Then all maps from the new family $\mathcal{F}^* = \{\kappa f : f \in \mathcal{F}_M\}$ satisfy $\mathfrak{B}_s(f) \leq 2\mathfrak{B}_e(f) < b(n,K)$, and thus by (i) $\mathcal{F}^*$ is equicontinuous with respect to the spherical metric.[2] Now we fix a compact $E \subset B(0,1)$ and $\epsilon \in (0,\pi)$. We choose $\delta > 0$ such that for $x, y \in E$ from $|x-y| < \delta$ follows $\text{dist}_s(f(x), f(y)) < \epsilon$ for every $f \in \mathcal{F}^*$. Let $f \in \mathcal{F}^*$. Then $g = f - f(x) \in \mathcal{F}^*$ because addition of a constant changes neither the Bloch radius nor K. So we have $\text{dist}_s(g(x), g(y)) = \text{dist}_s(0, g(y)) < \epsilon$ that is $|f(x) - f(y)| = |g(y)| \leq \tan(\epsilon/2)$. So for the members of the original family $\mathcal{F}_M$ we obtain $|f(x) - f(y)| < (3M/b(n,K))\tan(\epsilon/2)$. This proves equicontinuity with respect to the Euclidean metric.

Now (iv) follows from (iii) exactly like (ii) follows from (i). □

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