Herbert Stahl's proof of the BMV conjecture

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Theorem. *Let A and B be two $n \times n$ Hermitian matrices, where B is positive semi-definite. Then the function*

$$f(t) = \text{Tr } e^{A-tB} \tag{1}$$

is absolutely monotone, that is

$$f(t) = \int_0^\infty e^{-st} d\mu(s), \tag{2}$$

where μ is a non-negative measure.

This was conjectured in [1]. An equivalent statement is that the polynomial $t \mapsto \text{Tr}(A + Bt)^p,\ p \in \mathbf{N}$, has all non-negative coefficients, and that the function $t \mapsto \text{Tr}(A + tB)^{-p},\ p \geq 0$ is absolutely monotone, [2]. Theorem 1 was known for 2×2 matrices. The proof of Stahl, which is explained in these notes, is completely elementary: all tools that are used were available in the middle of XIX century.

Without loss of generality, one can assume that B is a diagonal matrix with eigenvalues $b_n > b_{n-1} > \ldots > b_1 > 0$. This is achieved by simultaneous conjugacy of A and B, and by approximating B with a matrix with distinct positive eigenvalues.

Now eigenvalues λ of $A - tB$ are determined from the equation

$$\det(\lambda I - A + tB) = 0.$$

This determinant is a polynomial in two variables t, λ. We take t out of the determinant, and denote $y = \lambda/t,\ x = 1/t$, then we obtain a polynomial equation of the form

$$0 = \det(yI + B - xA) = \prod_{j=1}^{n} (y + b_j - xa_{j,j}) + O(x^2),$$

where $O(x^2)$ is a polynomial divisible by x^2.

This implies that there are n holomorphic branches of the multivalued implicit function $\lambda(t)$ in a neighborhood of infinity, which satisfy

$$\lambda_j(t) = -b_j t + a_{j,j} + O(1/t), \quad t \to \infty, \tag{3}$$

and all λ_j are real on the real line. Moreover, each of these branches has an analytic continuation in a region containing the real line, according to Rellich's theorem [3, Thm XII.3]. The algebraic function $\lambda(t)$ is defined on a Riemann surface S with n sheets spread over the Riemann sphere. This Riemann surface is not necessarily connected. It has n unramified sheets over a region that contains the real line and a neighborhood of infinity.

Special case. Suppose that A is also diagonal, then the $O(1/t)$ terms in (3) can be omitted, and we obtain

$$f(t) = \sum_{j=1}^{n} e^{a_{j,j}} e^{-b_j t} = \int_0^\infty e^{-st} \sum_{j=1}^{n} e^{a_{j,j}} \delta_{b_j}(s) ds. \tag{4}$$

Thus μ is a discrete measure with positive atoms at the eigenvalues of B.

In the general case, the discrete component of μ is the same, and the continuous component is a positive function on (b_1, b_n).

Stahl figured out the following explicit expression for the density.

Proposition 1. *The measure*

$$d\mu(s) = \left(\sum_{j=1}^{n} e^{a_{j,j}} \delta_{b_j}(s) + w(s) \right) ds \tag{5}$$

where

$$w(s) = -\sum_{j:b_j<s} \mathrm{res}_\infty e^{\lambda_j(\zeta)+s\zeta} = \frac{1}{2\pi i} \sum_{j:b_j<s} \int_C e^{\lambda_j(\zeta)+s\zeta} d\zeta, \tag{6}$$

satisfies (1) and (2). Here C is any circle centered at the origin, of sufficiently large radius, described counterclockwise.

Lemma 1. *For every s, we have*

$$\sum_{j=1}^{n} \int_C e^{\lambda_j(\zeta)+s\zeta} d\zeta = 0.$$

Indeed, this is an integral of an entire function over a closed contour.

It follows that the density w defined by (6) is zero for $s > b_n$, and it is evidently zero for $s < b_1$.

Proof of Proposition 1. We compute the Laplace transform of the density w defined by (6).

$$\int_0^\infty e^{-st} w(s)ds = \sum_{k=1}^{n-1} \int_{b_k}^{b_{k+1}} e^{-ts} w(s)ds =: \sum_{k=1}^{n-1} I_k(t).$$

We fix $t > 0$ and deform the contour C in (6) so that the positive ray is outside C. Thus t is outside the deformed contour C'. According to (6), we have

$$I_k(t) = \int_{b_k}^{b_{k+1}} \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)+s(\zeta-t)} d\zeta ds.$$

By changing the order of integration and the order of summation, we obtain

$$\begin{aligned}\sum_{k=1}^{n-1} I_k(t) &= \sum_{j=1}^{n-1} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)} \int_{b_j}^{b_n} e^{s(\zeta-t)} ds d\zeta \\ &= \sum_{j=1}^{n-1} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)} \left(e^{b_n(\zeta-t)} - e^{b_j(\zeta-t)} \right) \frac{d\zeta}{\zeta - t}.\end{aligned}$$

The last expression is transformed using Cauchy's formula and the fact that t is outside C'. We have

$$\sum_{j=1}^{n} \int_{C'} e^{\lambda_j(\zeta)} e^{b_n(\zeta-t)} \frac{d\zeta}{\zeta - t} = 0,$$

similarly to the Lemma above, so

$$\sum_{k=1}^{n-1} I_k(t) = -\sum_{j=1}^{n} \frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)+b_j(\zeta-t)} \frac{d\zeta}{\zeta - t}.$$

Using (3), we write $\lambda_j(\zeta) = -b_j\zeta + a_{j,j} + r_j(\zeta)$, where $r_j(\infty) = 0$, and apply Cauchy's formula again. We obtain for every j:

$$\begin{aligned} - &\frac{1}{2\pi i} \int_{C'} e^{\lambda_j(\zeta)+b_j(\zeta-t)} \frac{d\zeta}{\zeta - t} = -\frac{e^{-b_j t + a_{j,j}}}{2\pi i} \int_{C'} e^{r_j(\zeta)} \frac{d\zeta}{\zeta - t} \\ = &\; e^{-b_j t + a_{j,j}} \left(e^{r_j(t)} - 1 \right) = e^{\lambda_j(t)} - e^{-b_j t + a_{j,j}}.\end{aligned}$$

Adding these expressions for $j = 1 \dots n$ and comparing with (5) and the second equation in (4), we obtain Proposition 1.

It remains to prove that (6) is non-negative for every s. Let us fix s and k so that $b_k < s < b_{k+1}$. The idea of Stahl, is to replace the contour of integration in (6) by ingeniously chosen homologous contour, on which the integral is non-negative simply because the integrand is non-negative!

We recall that S is a (possibly disconnected) Riemann surface spread over the ζ-sphere. We denote a generic point of S by p, and let $\pi : S \to \overline{\mathbf{C}}$ be the projection to ζ-plane. Then λ is a meromorphic function on S whose all poles are simple and lay over $\zeta = \infty$.

Asymptotic expressions (3) imply that there exists $R > 0$ such that for all $j \leq k$ the functions

$$\lambda_j(\zeta) + s\zeta = (s - b_j)\zeta + \dots \tag{7}$$

are holomorphic for $|\zeta| > R$, real on the real line and have strictly positive derivatives for $\zeta > R$ and $\zeta < -R$, while for $j > k$ they have strictly negative derivatives. By increasing R, if necessary, we achieve that for $|\zeta| > R/2$ and $j \leq k$, we have that $\operatorname{Im}(\lambda_j(\zeta) + s\zeta)$ has the same sign as $\operatorname{Im}\zeta$. And for $|\zeta| > R/2$ and $j > k$, $\operatorname{Im}(\lambda_j(\zeta) + s\zeta)$ has the opposite sign from $\operatorname{Im}\zeta$.

The surface S has an anti-conformal involution, induced by complex conjugation. The set of fixed points of this involution consists of n curves, π-preimages of the real line. These curves break S into two halves S^+ and S^- which are mapped onto each other by the involution. Projections of these halves are the upper and lower half-planes.

We set $C = \{\zeta : |\zeta| = R\}$ in (6), where R was just chosen.

Consider the open sets

$$D^+ := \{p \in S : |\pi(p)| < R,\ \operatorname{Im}\pi(p) > 0,\ \operatorname{Im}(\lambda(p) + s\pi(p)) > 0\},$$

$$D^- := \{p \in S : |\pi(p)| < R,\ \operatorname{Im}\pi(p) < 0,\ \operatorname{Im}(\lambda(p) + s\pi(p)) < 0\},$$

and

$$D = \operatorname{int}\left(\overline{D^+} \cup \overline{D^-}\right).$$

The set $\{p \in S : |\pi(p)| = R\}$ consists of n disjoint circles $C_j \subset S$ which we label according to the branches of λ_j in (7), so that $\lambda = \lambda_j$ on C_j. According to the paragraph after (7), the circles C_j with $j \leq k$ belong to ∂D while the C_j with $j > k$ are disjoint from $\overline{D}$.

Let D_1 be a component of D whose boundary contains some circles C_j.[1] We are going to prove that

$$\sum_{j:C_j\subset\partial D_1}\int_{C_j} e^{\lambda(p)+s\pi(p)}d\pi(p)>0, \tag{8}$$

where the circles are oriented counterclockwise, which agrees with their orientation as part of ∂D. Adding these relations over all components of D will prove the theorem. Indeed, each circle C_j with $j\leq k$ belongs to the boundary of exactly one component of D, and circles C_j with $j>k$ do not belong to the boundary of D.

Each component D_1 of D is a Riemann surface of finite type, whose boundary consists of several curves parametrized by circles. This parametrization is piecewise smooth, but may be neither smooth nor injective. We call these curves the boundary curves of D_1. Our choice of R guarantees that the part of the boundary of D_1 that projects in C is exactly the chain on which the integration is performed in (8). Consider the rest of the boundary ∂D_1 which projects into $|\zeta|<R$.

Lemma 2. *No boundary curve of D over $\{\zeta:|\zeta|<R\}$ can project into the open upper or lower half-plane.*

Indeed suppose that γ is a boundary curve whose projection does not intersect the real axis. It is oriented in the standard way, so that D is on the left. Suppose without loss of generality that γ projects to the upper half-plane. Let $g(p)=\lambda(p)+s\pi(p)$. As $\operatorname{Im} g>0$ in D^+, and $\operatorname{Im} g=0$ on γ, we conclude that the normal derivative of $\operatorname{Im} g$ has constant sign on γ. Then by the Cauchy-Riemann equations, the tangential derivative of $\operatorname{Re} g$ along γ is of constant sign, which is impossible because γ is a closed curve, and $\operatorname{Re} g$ is single valued on γ.

Thus every boundary curve of D_1 intersects the real line. Let γ be a boundary curve of D_1 which projects into $\{\zeta:|\zeta|<R\}$. By Lemma 2, γ is mapped into itself by the involution, so it consists of two symmetric pieces one piece γ^+ projects in the upper half-plane, another γ^- to the lower half-plane. At all endpoints p of γ^+ or γ^- we have $\Im\pi(p)=0$. We have

$$e^g=e^{\operatorname{Re} g+i\operatorname{Im} g}=e^{\operatorname{Re} g}\left(\cos(\operatorname{Im} g)+i\sin(\operatorname{Im} g)\right).$$

[1] One can prove using the maximum principle that every component of D has some C_j on the boundary, but we are not using this fact.

Since $\operatorname{Im} g = 0$ on γ, and $\operatorname{Re} g$ is increasing, we conclude that $\phi(t) := e^{g(\gamma(t))}$ is real and increasing function of the natural parameter t on γ^+. Thus

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma e^{g(p)}d\pi(p) &= \frac{1}{2\pi i}\left(\int_{\gamma^+}+\int_{\gamma^-}\right)\phi(t)(d\xi(t)+id\eta(t))\\ &= \frac{1}{\pi}\int_{\gamma^+}\phi(t)d\eta(t) = -\frac{1}{\pi}\int_{\gamma^+}\eta(t)d\phi(t) < 0,\end{aligned}$$

where we integrated by parts using $\eta(t) = 0$ on the endpoints of γ^+.

As the integral of the holomorphic 1-form over the boundary equals zero,

$$\int_{\partial D_1} e^g d\pi = 0$$

by Cauchy's theorem, the contribution to the integral from the part of ∂D_1 which projects to $\{\zeta : |\zeta| < R\}$ is the negative of the contribution of the part of ∂D_1 over C. This completes the proof of (8) and of Theorem 1.

References

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