# Julia directions for holomorphic curves

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**Abstract**

A theorem of Picard type is proved for entire holomorphic mappings into projective varieties. This theorem has local nature in the sense that the existence of Julia directions can be proved under natural additional assumptions. An example is given which shows that Borel's theorem on holomorphic curves omitting hyperplanes has no such local counterpart.

Let $\mathbf{P}^m$ be complex projective space of dimension $m$ and $M \subset \mathbf{P}^m$ be a projective variety. By a divisor on $M$ we mean an intersection of a hyperplane in $\mathbf{P}^m$ with $M$. We study holomorphic curves $f : \mathbf{C} \to M$.

**Theorem 1.** *Every holomorphic map* $f : \mathbf{C} \to M$, *omitting* $2n + 1$ *divisors such that any* $n + 1$ *of them have empty intersection, is constant.*

*Remark.* The dimension of $M$ is not mentioned in this formulation. Only the intersection pattern is relevant.

**Corollary.** *Every holomorphic map* $\mathbf{C} \to \mathbf{P}^n$, *omitting* $2n+1$ *hypersurfaces, such that any* $n + 1$ *of them have empty intersection, is constant.*

This Corollary also follows from the results of M. Green [4] and V. F. Babets [1]. Their proofs were based on Borel's theorem (which we will state later). We start with a simple proof of Theorem 1, independent of Borel's theorem. The method of the proof first appeared in [3]. It also provides a new proof of

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the classical Picard theorem [7, 9] as well as its generalizations to quasiregular maps in $\mathbf{R}^n$ [2, 7, 5].

*Proof of Theorem 1.* Let $P_1, \ldots, P_{2n+1}$ be the linear forms in $m+1$ variables defining the divisors. Consider a homogeneous representation $F = (f_0 : \ldots : f_m)$ of the curve $f$, where $f_j$ are entire functions without common zeros. Define the subharmonic function

$$u = \log ||F|| = \frac{1}{2} \log(|f_0|^2 + \ldots + |f_m|^2).$$

Suppose that $f$ is not constant. Then we may assume that the Riesz measure[1] $\mu$ of $u$ is infinite (if this is not the case, we can replace $f$ by $f \circ g$ with any transcendental entire $g$).

The functions

$$u_j = \log |P_j \circ F| = \log |P_j(f_0, \ldots, f_m)|, \quad j = 1, \ldots 2n+1,$$

are harmonic in $\mathbf{C}$.

Let $I \subset \{1, \ldots, 2n+1\}$, $\operatorname{card} I = n+1$. Let $\pi : \mathbf{C}^{m+1} \to \mathbf{P}^m$ be the standard projection. If $z \in \mathbf{C}^{m+1}$, $||z|| = 1$ and $\pi(z) \in M$ then for some constants $C_1$ and $C_2$ we have

$$C_1 \leq \max_{j \in I} |P_j(z)| \leq C_2$$

This follows from the assumption that the intersection of any $n+1$ divisors is empty. Using the homogeneity we conclude that

$$C_2 ||F(z)|| \leq \max_{j \in I} |P_j \circ F(z)| \leq C_2 ||F(z)||, \quad z \in \mathbf{C},$$

so

$$\max_{j \in I} u_j = u + O(1), \quad \operatorname{card} I = n+1. \tag{1}$$

In particular

$$\max_{1 \leq j \leq 2n+1} u_j = u + O(1). \tag{2}$$

We use the notation $D(a, r) = \{z \in \mathbf{C} : |z - a| < r\}$.

[1] We call it Cartan measure of $f$. Notice the formula $T(r, f) = \int_0^r \mu(D(0,t)) dt/t$.

**Lemma.** *Let $\mu$ be a Borel measure in $\mathbf{C}$, $\mu(\mathbf{C}) = \infty$. Then there exist sequences $a_k \in \mathbf{C}$, $a_k \to \infty$ and $r_k > 0$ such that*

$$M_k = \mu(D(a_k, r_k)) \to \infty \tag{3}$$

*and*

$$\mu(D(a_k, 2r_k)) \leq 200\mu(D(a_k, r_k)). \tag{4}$$

This Lemma is due to S. Rickman [10]. His formulation contains a minor mistake (see the discussion below). The Lemma was also used in [2]. In the end of the paper we will prove the lemma for completeness.

Apply the Lemma to the Riesz measure $\mu$ of the function $u$. We obtain two sequences $a_k$ and $r_k$, such that (3) and (4) are satisfied. Consider the functions defined in $D(0, 2)$:

$$u_k(z) = \frac{1}{M_k}(u(a_k + r_k z) - \tilde{u}(a_k + r_k z))$$

and

$$u_{j,k}(z) = \frac{1}{M_k}(u_j(a_k + r_k z) - \tilde{u}(a_k + r_k z)), \quad 1 \leq j \leq 2n + 1,$$

where $\tilde{u}$ is the smallest harmonic majorant of $u$ in the disc $D(a_k, 2r_k)$. The functions $u_k$ are Green potentials that is

$$u_k(z) = -\int_{D(0,2)} G(z, .)\, d\mu_k,$$

where $G(z, .)$ is the Green function of $D(0, 2)$ with pole at the point $z$ and $\mu_k$ is the Riesz measure of $u_k$.

It follows from (4) that $\mu_k(D(0, 2)) \leq 200$ so after selecting a subsequence we may assume that $u_k \to v$, where $v$ is a subharmonic function, not identically equal to $-\infty$. (Convergence holds in $L^1_{\text{loc}}(D(0, 2), dxdy)$, and the Riesz measures converge weakly, see [6, Theorem 4.1.9]). In particular $v$ is *not harmonic* because the Riesz measure of $\overline{D}(0, 1)$ is at least one in view of (3).

All functions $u_{j,k}$ are harmonic and bounded from above in view of (2), so we may assume that $u_{j,k} \to v_j$, each $v_j$ being harmonic or identically equal to $-\infty$ in $D(0, 2)$. From (1) and (3) follows

$$\max_{j \in I} v_j = v, \quad \text{card}\, I \geq n + 1. \tag{5}$$

Thus $v$ is continuous. For every $I \subset \{1, \ldots, 2n+1\}$ of cardinality $n+1$ we consider the set $E_I = \{z \in D(0,2) : v(z) = v_j(z),\ j \in I\}$. From (5) follows that the union of these sets coinsides with $D(0,2)$.

We conclude that at least one set $E_{I_0}$ has positive area. By the uniqueness theorem for harmonic functions all functions $v_j$ for $j \in I_0$ are equal. Applying (5) to $I_0$ we conclude that $v$ is harmonic. This is a contradiction which proves the theorem.

**Circles de remplissage and Julia directions**. In Rickman's formulation of the Lemma there is an extra property

$$r_k/|a_k| \to 0, \quad k \to \infty. \tag{6}$$

The following example shows that this property is not granted in general. Take

$$\mu(E) = \int_E \frac{dxdy}{x^2+y^2}, \quad E \subset \mathbf{C}.$$

Then all annuli $\{z : 2^m \leq |z| \leq 2^{m+1}\}$ have equal measure and we cannot find a sequence of discs $D(a_k, r_k)$ satisfying (3) and (6). However the following is true.

(*)*Let $\mu$ be a measure in $\mathbf{C}$ such that the sequence*

$$A_m = \mu(\{z : t^m \leq |z| \leq t^{m+1}\})$$

*is unbounded for some $t > 1$. Then there exist sequences $a_k \to \infty$ and $r_k > 0$ such that the conditions* (4)*,* (3) *and* (6) *are satisfied.*

It is clear that the assumption that $A_m$ is unbounded does not depend on $t > 1$. To prove (*) we pick a sequence $(m)$ of natural numbers such that $A_m \to \infty$. There is a covering of the annulus $\{z : t^m \leq |z| \leq t^{m+1}\}$ by $N_m \to \infty$ discs such that at least one of these discs, say $D_m$, still has large measure and satisfies (6). Now take this disc $D_m$ in place of $D(0, R/4)$ in the proof of the Lemma (see Appendix at the end of the paper). This proof will give us a disc $D(a_m, r_m)$ which satisfies (4), (3) and (6).

A number $\theta \in [0, 2\pi)$ is called a *Julia direction* for a holomorphic curve $f : \mathbf{C} \to M$ if for every system of divisors $D_1, \ldots, D_q$ such that any $n+1$ of them have empty intersection, and for any $\epsilon > 0$ all but at most $2n$ of these divisors have infinitely many preimages in the angle $\{z : |\arg z - \theta| < \epsilon\}$.

Using the statement (*) instead the Lemma we prove the following:

**Theorem 2.** *If the Riesz measure* $\mu$ *corresponding to a holomorphic curve* $f$ *has the property*

$$\limsup_{m\to\infty} \mu(\{z : 2^m \leq |z| \leq 2^{m+1}\} \to \infty$$

*then* $f$ *has at least one Julia direction.*

Actually under the assumptions of this theorem there exists a sequence of discs $D(a_k, r_k)$ satisfying (6) such that in the union of these discs all but at most $2n$ divisors have infinitely many preimages. Any accumulation point of the set $\{\arg a_k\}$ is a Julia direction. Such sequence of discs is called "circles de remplissage".

The condition on the measure $\mu$ in Theorem 2 is best possible. Actually there is an explicit description of all meromorphic functions (that is holomorphic curves in $\mathbf{P}^1$) having no Julia directions, which is due to A. Ostrovski (see, for example [8]).

The classical theorem of E. Borel can be formulated in the following way:

*Let* $f : \mathbf{C} \to \mathbf{P}^n$ *be a meromorphic curve with linearly independent components i. e., whose image is not contained in a hyperplane. Let* $L_1, \ldots, L_{n+2}$ *be hyperplanes in general position. Then* $f$ *cannot omit* $\cup_j L_j$.

In the following example a curve $f : \mathbf{C} \to \mathbf{P}^n$ with linearly independent components omits locally $2n$ hyperplanes in general position. That is there exists a covering of the plane by a finite set of angular sectors such that $f$ omits $2n$ hyperplanes in each sector of the set. So there is no analogue of Julia directions for Borel's theorem and the estimate $2n$ for the number of exceptional divisors is best possible even in the case when $M = \mathbf{P}^n$ and $f$ is linearly non-degenerate.

**Example.** For simplicity we construct the example only for $n = 2$. The coordinate representation of $f$ is $(f_0 : f_1 : f_2)$ where $f_j(z) = \sin(\varepsilon^j z)$, $\varepsilon = \exp(2\pi i/3)$, $j = 0, 1, 2$. The hyperplanes are defined by the vectors

$$(1, 0, 1)\,, (1, 0, 2)\,, (1, 1, 0)\,, (1, 2, 0)\,, (0, 1, 1)\,, (0, 1, 2).$$

A direct computation (or drawing a picture) shows that the system is admissible. In the angle $0 < \arg z < \pi/3$, we have $f_2 = o(f_0)$ and $f_2 = o(f_1)$ as

$z \to \infty$. So the hyperplanes defined by $(1,0,1), (1,0,2), (0,1,1)$ and $(0,1,2)$, are omitted in this angle. In the angle $\pi/4 < \arg z < 3\pi/4$ we have $f_2 = o(f_0)$ and $f_1 = o(f_0)$, so the hyperplanes defined by $(1,0,1), (1,0,2), (1,1,0)$ and $(1,2,0)$ are omitted in this angle. The other angles are studied similarly using the property that $f(\varepsilon z)$ is obtained from $f(z)$ by permutation of coordinates.

**Appendix**. *Proof of the Lemma.* Take a large number $R > 0$, so that $\mu(D(0,R/4))$ is large. Denote $\delta(z) = (R - |z|)/4$. Then find such $a \in D(0,R)$ that

$$\mu(D(a,\delta(a)) > \frac{1}{2} \sup_{z \in D(0,R)} \mu(D(z,\delta(z))). \tag{7}$$

Take $r = \delta(a)$. Then the disc $D(a,2r)$ can be covered by at most 100 discs of the form $D(z,\delta(z))$, so by (7)

$$\mu(D(a,2r)) \leq 200\mu(D(a,r)).$$

Putting $z = 0$ in (7) we get

$$\mu(D(a,r)) \geq \frac{1}{2}\mu(D(0,R/4)).$$

So we have constructed the disc of arbitrarily large measure and property (4). This proves the lemma.

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