Brody curves omitting hyperplanes

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September 4, 2009

Abstract

A *Brody curve*, a.k.a. normal curve, is a holomorphic map f from the complex line $\mathbf{C}$ to the complex projective space $\mathbf{P}^n$ such that the family of its translations $\{z \mapsto f(z+a) : a \in \mathbf{C}\}$ is normal. We prove that Brody curves omitting n hyperplanes in general position have growth order at most one, normal type. This generalizes a result of Clunie and Hayman who proved it for $n = 1$.

MSC 32Q99, 30D15.

Introduction

We consider holomorphic curves $f : \mathbf{C} \to \mathbf{P}^n$. The spherical derivative $\|f'\|$ measures the length distortion from the Euclidean metric in $\mathbf{C}$ to the Fubini–Study metric in $\mathbf{P}^n$. The explicit expression is

$$\|f'\|^2 = \|f\|^{-4} \sum_{i \neq j} |f_i' f_j - f_i f_j'|^2,$$

where $(f_0, \ldots, f_n)$ is a homogeneous representation of f (that is the f_j are entire functions which never simultaneously vanish), and

$$\|f\|^2 = \sum_{j=0}^{n} |f_j|^2.$$

A holomorphic curve is called a Brody curve if its spherical derivative is bounded. This is equivalent to normality of the family of translations $\{z \mapsto f(z+a) : a \in \mathbf{C}\}$.

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*Supported by NSF grant DMS-0555279 and by the Humboldt Foundation.

Brody curves are important for at least two reasons. First one is the rescaling trick known as Zalcman's lemma or Brody's lemma: for every non-constant holomorphic curve f one can find a sequence of affine maps $a_k : \mathbf{C} \to \mathbf{C}$ such that the limit $f \circ a_k$ exists and is a non-constant Brody curve. Second reason is Gromov's theory of mean dimension [4] in which a space of Brody curves is one of the main examples.

For the recent work on Brody curves we refer to [3, 10, 11, 12, 13]. A general reference for holomorphic curves is [6].

We recall that the Nevanlinna characteristic is defined by

$$T(r, f) = \int_0^r \frac{dt}{t} \left(\frac{1}{\pi} \int_{|z| \leq t} \|f'\|^2(z) dm_z \right),$$

where dm is the area element in $\mathbf{C}$. So Brody curves have order at most two normal type, that is

$$T(r, f) = O(r^2). \tag{1}$$

Clunie and Hayman [2] found that Brody curves $\mathbf{C} \to \mathbf{P}^1$ omitting one point in $\mathbf{P}^1$ must have smaller order of growth:

$$T(r, f) = O(r). \tag{2}$$

A different proof of this fact is due to Pommerenke [8]. In this paper we prove that this phenomenon persists in all dimensions.

Theorem. *Brody curves $f : \mathbf{C} \to \mathbf{P}^n$ omitting n hyperplanes in general position satisfy* (2).

Under the stronger assumption that a Brody curve omits $n+1$ hyperplanes in general position, the same conclusion was obtained by Berteloot and Duval [1] and Tsukamoto [11], with different proofs.

Combined with a result of Tsukamoto [10] our theorem implies

Corollary. *Mean dimension in the sense of Gromov of the space of Brody curves in*

$$\mathbf{P}^n \backslash \{n \text{ hyperplanes in general position}\}$$

is zero.

The condition that n hyperplanes are omitted is exact: it is easy to show by direct computation that the curve $(f_0, f_1, 1, \ldots, 1)$, where f_i are appropriately chosen entire functions such that f_1/f_0 is an elliptic function, is a Brody curve, it omits $n-1$ hyperplanes, and $T(r,f)\sim cr^2,\ r\to\infty$ where $c>0$. This example will be discussed in the end of the paper.

The author thanks Alexandr Rashkovskii and Masaki Tsukamoto for inspiring conversations on the subject.

Preliminaries

Without loss of generality we assume that the omitted hyperplanes are given in the homogeneous coordinates by the equations $\{w_j=0\},\ 1\leq j\leq n$. We fix a homogeneous representation $(f_0,\ldots,f_n)$ of our curve, where f_j are entire functions without common zeros, and $f_n=1$. We assume without loss of generality that $f_0(0)\neq 0$.

Then

$$u=\log\sqrt{|f_0|^2+\ldots+|f_n|^2} \tag{3}$$

is a positive subharmonic function, and Jensen's formula gives

$$T(r,f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(re^{i\theta})d\theta-u(0)=\int_0^r\frac{n(t)}{t}dt,$$

where $n(t)=\mu(\{z:|z|\leq t\})$, and μ is the Riesz measure of u, that is the measure with the density

$$\frac{1}{2\pi}\Delta u=\frac{1}{\pi}\|f'\|^2. \tag{4}$$

Now positivity of u and (1) imply that all f_j are of order at most 2, normal type.

In particular,

$$f_j=e^{P_j},\quad 1\leq j\leq n,$$

where P_j are polynomials of degree at most two.

First we state a lemma which is the core of our arguments. It is a refined version of Lemma 1 in [2]. We denote by $B(a,r)$ the open disc of radius r centered at the point a.

Lemma 1. *Let u be a non-negative harmonic function in the closure of the disc $B(a,R)$, and assume that $u(z_1)=0$ for some point $z_1\in\partial B(a,R)$. Then*

$$|\nabla u(z_1)|\geq\frac{u(a)}{2R}.$$

Proof. The function

$$b(r)=\min_{|z-a|=r}u(z)$$

is decreasing and $b(R)=0$. Harnack's inequality gives

$$b(t)\geq\frac{R-t}{R+t}u(a),\quad 0\leq t\leq R.$$

As

$$b(t)=|b(R)-b(t)|\leq(R-t)\max_{[t,R]}|b'|,$$

we conclude that for every $t\in(0,R)$ there exists $r\in[t,R]$ such that

$$|b'(r)|\geq\frac{1}{R-t}\frac{R-t}{R+t}u(a)=\frac{u(a)}{R+t}.$$

According to Hadamard's three circle theorem, $rb'(r)$ is a negative decreasing function, so

$$|Rb'(R)|\geq|rb'(r)|\geq r\frac{u(a)}{R+t}\geq t\frac{u(a)}{R+t},$$

and the last expression tends to $u(a)/2$ as $t\to R$. So we have $|b'(R)|\geq u(a)/(2R)$. On the other hand, $|\nabla u(z_1)|\geq\left|\frac{du}{dn}(z_1)\right|\geq|b'(R)|$, where d/dn is the normal derivative. This completes the proof.

Proof of the theorem

We may assume without loss of generality that f_0 has at least one zero. Indeed, we can compose f with an automorphism of $\mathbf{P}^n$, for example replace f_0 by f_0+cf_1, $c\in\mathbf{C}$ and leave all other f_j unchanged. This transformation changes neither the n omitted hyperplanes nor the rate of growth of $T(r,f)$ and multiplies the spherical derivative by a bounded factor.

Put $u_j=\log|f_j|$, and

$$u^*=\max_{1\leq j\leq n}u_j.$$

Here and in what follows max denotes the pointwise maximum of subharmonic functions. We are going to prove first that

$$u_0(z)\leq u^*(z)+4(n+1)|z|\sup_{\mathbf{C}}\|f'\|. \tag{5}$$

for $|z|$ sufficiently large.

Let a be a point such that $u_0(a) > u^*(a)$. Consider the maximal disc $B(a,R)$ centered at a where the inequality $u_0(z) > u^*(z)$ still holds. If z_0 is a zero of f_0 then $u_0(z_0) = -\infty$ and we have

$$R \leq |a| + |z_0| \leq 2|a|, \tag{6}$$

for $|a| > |z_0|$. There is a point $z_1 \in \partial B(a,R)$ and an integer $k \in \{1,\ldots,n\}$ such that

$$u_0(z_1) = u^*(z_1) = u_k(z_1) \geq u_j(z_1), \tag{7}$$

for all $j \in \{1,\ldots,n\}$. Applying Lemma 1 to the positive harmonic function $u_0 - u_k$ in $B(a,R)$ we obtain

$$|\nabla(u_0 - u_k)(z_1)| \geq \frac{u_0(a) - u_k(a)}{2R},$$

or

$$u_0(a) \leq u_k(a) + 2R\,|\nabla u_0(z_1) - \nabla u_k(z_1)|\,. \tag{8}$$

On the other hand, $|f_0(z_1)| = |f_k(z_1)| \geq |f_j(z_1)|$ for all $j \in \{1,\ldots,n\}$, so

$$\|f'(z_1)\| \geq \frac{|f_0'(z_1)f_k(z_1) - f_0(z_1)f_k'(z_1)|}{|f_0(z_1)|^2 + \ldots + |f_n(z_1)|^2} \geq (n+1)^{-1}\left|\frac{f_0'(z_1)}{f_0(z_1)} - \frac{f_k'(z_1)}{f_k(z_1)}\right|. \tag{9}$$

Combining (8), (9) and (6), and taking into account that $|\nabla \log|f|| = |f'/f|$, we obtain (5).

If all polynomials P_j are linear then inequality (5) completes the proof. Suppose now that some P_j is of degree 2.

Consider again the subharmonic functions $u_j = \log|f_j|,\ 0 \leq j \leq n$. For each $j \in \{0,\ldots,n\}$, the family

$$\{r^{-2}u_j(rz) : r > 1\}$$

in uniformly bounded from above on compact subsets of the plane, and bounded from below at 0. By [5, Theorem 4.1.9] these families are normal (from every sequence one can choose a subsequence that converges in L^1_{loc}). Take a sequence r_k such that

$$\lim_{k\to\infty} \frac{1}{r_k^2}\int_{-\pi}^{\pi} u(r_k e^{i\theta})d\theta > 0, \tag{10}$$

where u is defined in (3). Such sequence exists because we assume that at least one of the P_j is of degree two.

Then we choose a subsequence (still denoted by r_k) such that

$$r_k^{-2} u_j(r_k z) \to v_j, \quad 0 \leq j \leq n,$$

and $r_k^{-2} u(r_k z) \to v$, where v_j, v are some subharmonic functions in $\mathbf{C}$. Then

$$v = \max\{v_0, \ldots, v_n\} \neq 0$$

is a non-negative subharmonic function. Let ν be the Riesz measure of v. Notice that $\nu \neq 0$ because v is non-negative and $v \neq 0$. We have weak convergence

$$\nu = \lim_{k \to \infty} \mu_{r_k},$$

where

$$\mu_{r_k}(E) = r_k^{-2} \mu(r_k E)$$

for every Borel set E. Now (4) and the condition that $\|f'\|$ is bounded imply

Lemma 2. *ν is absolutely continuous with respect to Lebesque's measure in the plane, with bounded density.*

Proof. For every disc $B(a, \delta)$ we have

$$\nu(B(a, \delta)) \leq \liminf_{k \to \infty} r_k^{-2} \mu(B(r_k a, r_k \delta)) \leq \delta^2 \sup_{\mathbf{C}} \|f'\|^2.$$

Now we invoke our inequality (5). It implies that

$$v_0 \leq v^* = \max(v_1, \ldots, v_n),$$

so $v = v^*$. Thus the measure ν is supported by finitely many rays. This contradiction with Lemma 2 shows that all polynomials P_j are in fact linear. This completes the proof.

Example

Let $\Gamma_0 = \{n + im : n, m \in \mathbf{Z}\}$ be the integer lattice in the plane, and $\Gamma_1 = \Gamma + (1 + i)/2$. For $j \in \{0, 1\}$, let f_j be the Weierstrass canonical products of genus 2 with simple zeros on Γ_j. Then the f_j are entire functions of completely regular growth in the sense of Levin–Pfluger and their zeros satisfy the R-condition in [7, Theorem 5, Ch. 2]. This theorem of Levin implies that

$$\log|f_j(re^{i\theta})| = (c+o(1))r^2, \tag{11}$$

as $r\to\infty$, $re^{i\theta}\notin C_0$ where C_0 is a union of discs of radius 1/4 centered at the zeros of f_j. It follows that

$$|f_0(z)|^2+|f_1(z)|^2\to\infty, \quad z\to\infty. \tag{12}$$

Cauchy's estimate for the derivative and (11) give

$$\log|f_j'(z)|\leq (c+o(1))|z|^2, \quad z\to\infty.$$

So for the curve $f=(f_0,f_1,1,\ldots,1)$ we obtain

$$\begin{aligned}\|f'\|^2 &= \frac{\sum_{i\neq j}|f_i'f_j-f_if_j'|^2}{\|f\|^4}\leq\frac{(|f_0'f_1-f_0f_1'|^2+n(|f_0'|^2+|f_1'|^2))}{(|f_0|^2+|f_1|^2)^2}\\ &= \frac{|g'|^2}{(1+|g|^2)^2}+o(1).\end{aligned}$$

The spherical derivative of g is bounded because g is an elliptic function. Thus f is a Brody curve that omits $n-1$ hyperplanes in general position. Evidently $T(r,f)\sim c_1r^2$.

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