More on residues. Bürmann–Lagrange formula

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Let

$$f = \sum_{n=-m}^{\infty} a_n z^n \tag{1}$$

be a *formal* Laurent series. (Possibly not convergent anywhere). I recall that such series can be added and multiplied without any reference to convergence. They can be also differentiated by the usual rules. We cannot plug a complex number z into such series, except when $m \geq 0$ and $z = 0$. Notice that we will never plug anything else in what follows!

We define the residue as

$$\text{res}\, f = a_{-1}.$$

Notice that $\text{res}\, f' = 0$ for every series of the form (1). It follows that

$$\text{res}\,(fg') = \text{res}\,(fg)' - \text{res}\,(f'g) = -\text{res}\,(f'g), \tag{2}$$

which resembles the "integration by parts formula". The usual formula

$$\text{res}\, f = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z^m f)|_{z=0} \tag{3}$$

also holds.

Now consider a formal Taylor series which begins with $c_1 z$:

$$\phi(z) = \sum_{n=1}^{\infty} c_n z^n, \quad c_1 \neq 0. \tag{4}$$

Notice that *composition* $f \circ \phi$ of two series of the forms (1) and (4) can be defined: if we substitute one into another, the coefficients of the resulting series can be computed without any consideration of convergence: they are *finite* combinations of the coefficients of the two original series. Verify this. (Composition of two arbitrary formal series of the form (1) *cannot be defined*!)

Next, prove that the set of all series of the form (4) is a *group* with respect to composition. That is the composition is associative and every series of the form (4) has an inverse. The neutral element of this group is of course the series $f(z) = z$.

Our goal is to obtain an explicit formula for the coefficients of the compositional inverse. Our main tool is the following "change of the variable" property:

$$\text{res}\,[(f \circ \phi)\phi'] = \text{res}\, f. \tag{5}$$

Prove this! Of course, if the series were convergent, you could use the relation with the integral

$$\text{res}\, f = \frac{1}{2\pi i} \int_{|z|=\epsilon} f(z) dz.$$

Then our formula woulod follow follows by the change of the variable in this integral. However (5) is true and is easy to verify for all formal series, and I recommend that you verify it by an algebraic manipulation.

Formula (5) suggests that the residue is defined for a *differential* rather than a function, so it is better to write res $[f(z)dz]$.

Now suppose that ϕ is given by (4), and

$$\phi^{-1} = \sum_{n=1}^{\infty} b_n w^n,$$

and we want to find a formula for the b_n in terms of c_n. We write

$$b_n = \text{res}\, \frac{\phi^{-1}}{w^{n+1}} = \text{res}\, \left(z \frac{\phi'}{\phi^{n+1}} \right),$$

where we made the change of the variable $w = \phi(z)$ and used (5). Using the "integration by parts formula" (2) we can continue:

$$b_n = \text{res}\, \left(z \frac{\phi'}{\phi^{n+1}} \right) = \frac{1}{n} \text{res}\, \frac{1}{\phi^n},$$

and finally, using (3) we obtain

$$b_n = \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z^n}{\phi^n} \right) \Bigg|_{z=0}. \tag{6}$$

This is called the *Bürmann–Lagrange formula* for the coefficients of the inverse function.

Example. Let $\phi(z) = ze^{-z}$. Show that the branch of the inverse function that maps 0 to 0 is given by the series

$$\phi^{-1}(w) = \sum_{n=1}^{\infty} n^{n-1} \frac{z^n}{n!}.$$

Of course, all series are convergent in this example. There are actually few examples where the Bürmann–Lagrange formula gives a simple answer in closed form. However, for convergent series, it always gives a representation of the coefficients in the form of an integral, and this can be useful in the study of the asymptotic behaviour of these coefficients. But this is another story.

Example (Hurwitz). Expand the branch ϕ^{-1} which takes 0 to 0:

$$\phi(z) = (e^z - 1)e^{-z}.$$

Example (Hurwitz). Expand the branch ϕ^{-1} which takes 0 to a:

$$\phi(z) = 2\frac{z-a}{z^2-1}.$$

Example (MO) Solve the equation $z^5 - z + a = 0$ in the form of power series

$$z = -\sum_{k=0}^{\infty} \binom{5k}{k} \frac{a^{4k+1}}{4k+1}.$$

Solution. Write the equation in the form

$$\phi(z) := z - z^5 = a,$$

and apply (6) to $z = \phi^{-1}(a)$. The n-th coefficient of ϕ^{-1} will be

$$\left.\frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\frac{1}{(1-z^4)^n}\right|_{z=0}$$

which is equal to $1/n$ times the coefficient at z^{n-1} in the expansion

$$(1 + z^4 + z^8 + \ldots)^n.$$

This coefficient is different from 0 only if $n = 4k+1$, and is equal to the coefficient at w^k in the expansion of

$$\frac{1}{(1-w)^{4k+1}} = \frac{1}{(4k)!}\frac{d^{4k}}{dw^{4k}}\frac{1}{1-w}$$

$$= \frac{1}{(4k)!}\frac{d^{4k}}{dw^{4k}}(1+w+w^2+\ldots),$$

where we set $w = z^4$. So and we obtain that this coefficient is

$$\frac{1}{(4k)!}\frac{d^{4k}}{dw^{4k}}w^{5k} = \binom{5k}{k}.$$

So the $4k+1$-th coefficient of our function is

$$\frac{1}{4k+1}\binom{5k}{k}.$$

Exercise. Find the radius of convergence of this series.

The Burmann–Lagrange formula can be generalized: one can find the expansion of $f \circ \phi^{-1}$, where f is a given analytic function. Assuming $\phi(0) = 0, \phi'(0) \neq 0$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we denote by b_n the coefficients of $f \circ \phi^{-1}$. Then we have

$$b_n = \text{res}\left[\frac{f \circ \phi^{-1}(u)}{u^{n+1}}du\right] = \text{res}\left[\frac{f(z)}{\phi^{n+1}(z)}\phi'(z)dz\right].$$

"Integrating by parts", and applying the formula for the residue, we obtain

$$b_n = -\frac{1}{n}\text{res}\left[\phi dg^{-n}\right] = \frac{1}{n}\text{res}\left[g^{-n}d\phi\right]$$

$$b_n = \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\left(\frac{z}{\phi(z)}\right)^n f'(z)\right].$$

Example. Taking $\phi(z) = ze^{-\beta z}$ and $f(z) = e^{\alpha z}$, and putting $z = 1$, we obtain an interesting identity

$$\sum_{n=0}^{\infty} \alpha(\alpha + \beta n)^{n-1} e^{-(\alpha+\beta n)}/n! = 1,$$

which is valid for all α and $|\beta e^{-\beta}| < 1$. The sequence in the left hand side is called the Poisson-type distribution.

References

[1] https://mathoverflow.net/questions/32099/what-is-lagrange-inversion-good-for/32261#32261.