Brief story of the first transatlantic cable

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Until 1866, there was no way to send a message from England to America which would arrive in less than few weeks.

The first successful transatlantic cable was laid down in 1866. The story of this cable reads like a Jules Verne novel[1]. If I were on Jules Verne's place, I'd rather describe this story then invent similar fictional stories.

Besides exciting adventures, the story of the Cable has many scientific aspects. Let us consider a mathematical model of a long cable. I follow [4] but all this math is due to William Thomson[2] and George Stokes.

What should the parameters of such a long cable be? What happens to the transmitted signal?

Let x be the space coordinate, and t the time. Let $u(x,t)$ be the potential (voltage) and $i(x,t)$ the current at the point x at time t. Small piece Δx of the cable has some inductance $L\Delta x$, capacity $C\Delta x$, passive resistance $R\Delta x$, and baking conductance (conductance through the insulation) $G\Delta x$. The quantities L, C, R are measured in farad/meter, henry/meter, ohm/meter and 1/(ohm.meter), respectively.

Generalized Ohm's laws give

$$u(x,t) - u(x+\Delta x,t) = L\frac{\partial i}{\partial t}\Delta x + Ri\Delta x,$$

$$i(x,t) - i(x+\Delta x,t) = C\frac{\partial u}{\partial t}\Delta x + Gu\Delta x.$$

[1]One of the main characters of Jules Verne's novel "Mysterious island", engineer Cyrus Smith had a prototype: Cyrus Fields, the main character of the real Cable story.

[2]William Thomson, 1st baron Kelvin. A great XIX century mathematician, physicist, and engineer. He was made baron Kelvin by Queen Victoria for his contribution to this Transatlantic Cable enterprise.

Dividing by Δx, we obtain the following differential equations

$$\begin{cases} u_x + Li_t + Ri &= 0, \\ i_x + Cu_t + Gu &= 0. \end{cases}$$

To separate u and i, we apply $\partial/\partial x$ to the first equation, and $\partial/\partial t$ to the second one, and eliminate i and its derivatives using $i_{xt} = i_{tx}$, to obtain

$$u_{xx} = LCu_{tt} + (CR + GL)u_t + RGu = 0.$$

This is called the *Telegraph Equation*. It is some mixture of heat and wave equations.

In a good cable, L and G are negligible, so we obtain a heat equation

$$u_t = ku_{xx}, \quad \text{where} \quad k = \frac{1}{RC}. \tag{1}$$

Consider this equation on a half-line $x \geq 0$, with zero initial conditions and boundary condition $f(t)$:

$$u(0,t) = f(t), \quad u(x,0) = 0. \tag{2}$$

This describes a transmission of a signal $f(t)$ from an endpoint of a very long cable. Here $f(t)$ represents the signal which we are trying to transmit.

Equation (1) with initial and boundary conditions (2) can be exactly solved, see "Applications of Fourier Transform", section 3.

$$u(x,t) = \frac{x}{2\sqrt{\pi k}} \int_0^t f(t-s)s^{-3/2} \exp\left(-\frac{x^2}{4ks}\right) ds.$$

This is a convolution of f with the *lateral heat kernel*

$$L(x,t) := \frac{x}{2\sqrt{\pi k}} t^{-3/2} e^{-\frac{x^2}{2kt}}, \quad t > 0,$$

and $L(s,t) = 0$ for $t < 0$.

To understand the properties of this solution, let us introduce the universal (independent on any parameters) function

$$g(t) = t^{-3/2} e^{-1/t}. \tag{3}$$

and recall the scaling operation which does not change the integral: $g_\delta(t) := \delta^{-1} g(t/\delta)$. Then a straightforward calculation shows that

$$L(x,t) = \frac{1}{\sqrt{\pi}} g_\delta(t)$$

with

$$\delta = \frac{x^2}{4k} = \frac{1}{4} x^2 RC. \tag{4}$$

This means the following: suppose we want to transmit a short impulse, say some positive function $f(t)$ with support $(0, \epsilon)$. This signal arrives to the point x as a convolution $(1/\sqrt{\pi}) f \star g_\delta$, where g is the universal function (3) and δ is proportional to $x^2 RC$. So the signal that arrives to x has time duration

$$\sim \epsilon x^2 RC,$$

that is it is spread in time, and this spread grows quadratically with the distance. This is the famous "square law" of Thomson.

This means that you cannot transmit too quickly. It took many hours to transmit the few sentences in the first telegrams sent by the first transatlantic cable. You cannot do anything with the factor x^2 in δ; it is the distance to which you want to transmit the signal.

Only R and C are at your disposal, and they depend on the characteristics of the cable. I recall that R is passive resistance, per unit of length, and it is proportional to the cross-section area, that is to r^{-2}, where r is the radius of the cable. And C is capacity per unit of length which is proportional to r. So RC is proportional to $1/r$ and a thick cable will transmit faster.

The scientific advisers of the first cable project did not understand this; for some reasons they thought that capacity is more important and that thin cable will do the job. Of course the investors were very happy with this theory, since the cost increases dramatically with the thickness of the cable.

When they saw the frustratingly slow rate of transmission, they started to increase voltage... and burned the cable in few weeks after the start of the operation. And the whole enterprise was bankrupt.

The theory which I explained above belongs to William Thomson and George Stokes.

The enterprise was saved by the British government[3]. Thomson was appointed the new scientific adviser. With the money from British government

[3] During those few days that the first cable worked they managed to transmit one

the company hired the largest existing ship to lay a new cable. Thomson designed a super-sensitive galvanometer for receiving the messages. And the new cable was a great success. Thomson was knighted and since then he is known as Lord Kelvin.

The modern theory of cables, taking into account the full telegraph equation, not just its approximation (1) is due to Oliver Heaviside, an electric engineer who made very substantial contribution to mathematics. The heat equation gives a good approximation when we neglect inductance (see the derivation above), and this is OK when we deal with very low frequencies. For high frequencies, the telegraph equation resembles more the wave equation than the heat equation. Old telegraphs transmitted very low frequency signals, corresponding to dots and dashes of the Morse code entered manually.

Such low frequency signals are rarely used today. One exception is VLF (Very Low Frequency) systems which are used in communication with submarines. Water is impenetrable for higher frequency radio signals.

When we talk about "long cables", this is just a convenient terminology. I had a friend, a computer scientist, working for a company which designed computer chips. Once he started asking me about heat and telegraph equations, and from his questions I understood that he needs this "long cable" theory of Stokes, Kelvin and Heaviside. I was surprised: how are "long cables" related to computer chips design? He explained me that the "cable" has length of the order $10^{-2}mm$ but it is so thin that has to be considered "long".

It is remarkable that the authors of the article on Cable theory in Wikipedia apparently think that "Cable theory" is a part of neuroscience.

This demonstrates the universality of mathematical methods. Same mathematics describes transatlantic cables, microscopic conductors inside a chip and neurons in an animal brain.

References

[1] G. Folland, Fourier analysis and its applications, Brooks/Cole Publ., 1992.

telegram which resulted in financial savings much larger than the whole amount spent on laying the cable. This was a telegram cancelling relocation of an infantry regiment from Canada to India.

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[4] M. Lavrentiev, B. Shabat, Methiods of the theory of functions of a complex variable, Moscow, Nauka, 1973.

[5] William Thomson, On the theory of the electric telegraph, Proc. Royal Soc. London, 7 (1854-55) 382–399.