How complex numbers were discovered

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1. Let us recall the derivation of Cardano's formula for the cubic equation

$$x^3 = ax + b.$$

We set $x = u + v$:

$$u^3 + v^3 + 3uv(u + v) = a(u + v) + b.$$

We can impose one more relation on u, v:

$$3uv = a.$$

Now we obtain a system of equations for u^3, v^3:

$$\begin{aligned} u^3v^3 &= \frac{a^3}{27}, \\ u^3 + v^3 &= b. \end{aligned}$$

It follows (by Vieta's formulas) that u^3 and v^3 are two solutions of the quadratic equation

$$w^2 - bw + \frac{a^3}{27} = 0,$$

and thus

$$x = \sqrt[3]{u} + \sqrt[3]{v} = \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}.$$

This is Cardano's formula discovered in XVI century in Italy.

2. Now suppose that we don't know complex numbers. The formula makes sense only when $b^2/4 - a^3/27 \geq 0$. However, this condition holds if and only if our equation has one real root. Verify this! In this case, the formula gives the correct answer. For example, equation

$$x^3 - 2x - 4 = (x-2)(x^2 + 2x + 2) = 0$$

has one real root 2, and this root can be obtained by Cardano's formula. The last statement is non-trivial: one has to show that

$$\sqrt[3]{2 - \frac{10}{\sqrt{27}}} + \sqrt[3]{2 + \frac{10}{\sqrt{27}}} = 2.$$

Can you show this without a calculator?

What happens when $b^2/4 - a^3/27 < 0$? Let us take the simplest example with $a = 1$, $b = 0$, so the equation is

$$x^3 = x$$

which has roots $0, 1, -1$.

Cardano's formula gives

$$x = \sqrt[3]{-\sqrt{-1/27}} + \sqrt[3]{\sqrt{-1/27}} = \frac{1}{\sqrt{3}}(\sqrt[3]{-i} + \sqrt[3]{i}),$$

so we need complex numbers, even to find real solutions of a problem which involvs only real numbers!

Verify that this formula gives all three roots, $0, 1, -1$, and in addition 4 numbers which are not roots of our equation.

The following example belongs to Bombelli (1572). The equation

$$x^3 = 15x + 4$$

when solved with Cardano's formula gives

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}.$$

He found (by trial and error?) that

$$(2 \pm i)^3 = 2 \pm 11i,$$

so $x = 2+i+(2-i) = 4$. The other roots are $-2 \pm \sqrt{3}$. How does Cardano's formula give them?

All this puzzled mathematicians since XVI century. It took more than 200 years to resolve all these questions satisfactory and to define complex numbers rigorously. Even in XIX century many mathematicians were uneasy with them. They say that French students even made a riot once, in the early XIX century, because they were taught complex numbers.

Remark. You may conclude from these examples that Cardano's formula is not very useful for solving cubic equations in practice. This is correct. However this formula played a very important role in the development of mathematics:

a) Complex numbers had to be invented, and

b) After having found a similar formula for equations of 4-th degree, many efforts were spent on search of such a formula for equations of higher degree. Then it was proved that no such formula exists, but the proofs of this fact led to new fundamental mathematical theories which are central in modern mathematics. These theories are much more important than the problem for which they were invented.