Holomorphic curves omitting five planes in projective space

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Abstract

In 1928 H. Cartan proved an extension of Montel's normality criterion to holomorphic curves in complex projective plane $\mathbf{P}^2$. He also conjectured that a similar result is true for holomorphic curves in $\mathbf{P}^n$ for any n. Recently the author constructed a counterexample to this conjecture for any $n \geq 3$. In this paper we show how to modify Cartan's conjecture so that it becomes true, at least for $n = 3$.

1. Introduction. A classical theorem of Borel says that any holomorphic mapping $f : \mathbf{C} \to \mathbf{P}^n$ omitting $p = n + 2$ hyperplanes in general position must be linearly degenerate – that is the image $f(\mathbf{C})$ must be contained in a hyperplane. To state the theorem more precisely we choose the representation of $\mathbf{P}^n$ as the hyperplane in $\mathbf{P}^{n+1}$ defined in homogenous coordinates by the equation $x_0 + \ldots + x_{n+1} = 0$. This representation has the advantage that the $n + 2$ omitted hyperplanes can be described by symmetric equations $x_j = 0,\ 0 \leq j \leq n + 1$.

Borel's Theorem. *Let f_j be entire functions without zeros satisfying*

$$f_1 + \ldots + f_p = 0.$$

Then there exists a partition of the set of functions $\{f_j\}$ into classes such that all functions in the same class are constant multiples of each other and the sum of the functions in each class is zero.

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The case $p = 3$ of Borel's theorem is nothing but the Little Picard Theorem. Indeed, to say that an entire function f omits 0 and 1 is the same as to say $f + g - 1 = 0$, where f and g have no zeros.

According to the so-called Bloch Principle, to Borel's theorem there should correspond a normality criterion, just as Montel's theorem corresponds to Picard's theorem. We refer to [2, 6] and [7] for general discussion of this heuristic principle. But, as Bloch remarks in [1], cf. [6, p. 224] it is not at all clear at first sight what this normality criterion should be.

Set $D(a, r) = \{z \in \mathbf{C} : |z| < r\}$, $D(r) = D(0, r)$ and let U denotes the set of holomorphic functions g in $D(1)$ such that $g(z) \neq 0$, $z \in D(1)$. Such functions are called *units*. We are going to study infinite families $F = \{f\}$ of p-tuples $f = (f_1, \ldots, f_p)$, $f_j \in U$, satisfying the equation

$$f_1 + \ldots + f_p = 0. \tag{1}$$

Given such a family F let $\mathcal{F}$ denotes the filter formed by complements of finite subsets of F.

A subset of indices $S \subset \{1, \ldots, p\}$ is called a C-class if
(i) there exists $k \in S$ such that f_j/f_k are uniformly bounded on compacta as $f \to \mathcal{F}$ for all $j \in S$
and
(ii) $\sum_{j \in S} f_j/f_k \to 0$ as $f \to \mathcal{F}$, uniformly on compacta.

Notice that by (ii) every C-class contains at least 2 elements.

Cartan's conjecture. *Given a family F of p-tuples of units satisfying* (1) *there exists an infinite subfamily $L \subset F$ such that for $f \in L$ the set of indices $\{1, \ldots, p\}$ can be partitioned into C-classes.*

For $p = 3$ this is nothing else but Montel's normality criterion. The conjecture was stated by Cartan in [3], where he proved

Cartan's Theorem. *Given a family F of p-tuples of units satisfying* (1) *there exists an infinite subfamily $L \subset F$ such that for $f \in L$ one of the following holds:*
a) the full set of indices $\{1, \ldots, p\}$ forms a C-class
or
b) there exist at least two disjoint C-classes.

This theorem implies Cartan's conjecture for $p = 4$ because if the case b) occurs then two C-classes make a partition of the set of indices. When $p \geq 5$ Cartan's theorem falls short of proving his conjecture because in the case b) there may not be enough C-classes to make a partition of $\{1, \ldots, p\}$. On [3, pp. 69-70] Cartan discusses the hypothetical case when $p = 5$ and there are only two C-classes each containing two elements but the remaining index does not belong to any C-class. He concludes that constructing such an example would be difficult.

Such example has been recently constructed in [4]. A simplified version will be given in section 4. Actually this example shows that Cartan's conjecture fails even if we replace condition (ii) in the definition of C-class by a weaker condition that every C-class contains at least two elements. Examination of the example as well as our strong belief in Bloch's Principle suggest the following

Modified Conjecture. *Let F be an infinite family of p-tuples of units in $D(1)$ satisfying* (1). *Then there exists an infinite subfamily $L \subset F$ such that for $f \in L$ the set of indices can be partitioned into C-classes in the disk $D(r_p)$ where $0 < r_p < 1$ and r_p depends only on p.*

It will follow from this conjecture that in any hyperbolic disk of sufficiently small radius the partition of the set of indices into C-classes is possible.

We can prove this Modified Conjecture only for $p = 5$. The proof given in section 3 is based on the the same techniques used by Bloch and Cartan, that is Nevanlinna theory and estimates of potentials. A very good reference is [6]. The new ingredient is an elementary lemma from potential theory contained in section 2.

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2. An auxilliary result on harmonic functions

Lemma 1 *Let u_1 and u_2 be harmonic functions in the disk $D(z_0, r)$. Denote by u_+ the least harmonic majorant of $u_1 \vee u_2$ and by u_- the greatest harmonic minorant of $u_1 \wedge u_2$. If $u_+ \geq 0$ and $u_-(z_0) + \delta u_+(z_0) \geq 0$ for some δ, $0 < \delta < 1$, then one of the functions u_1, u_2 is non-negative in $D(z_0, ar)$, where*

$a = a(\delta)$ is given by

$$a(\delta) = \frac{\sqrt{2} - \sqrt{1+\delta}}{\sqrt{2} + \sqrt{1+\delta}}. \tag{2}$$

Furthermore, we actually have $u_i(z) \geq \epsilon u_+(z_0)$ for $i = 1$ or 2 in the disk $D(z_0, a'r)$, $a' < a(\delta)$, for some ϵ depending only on a'.

Remarks. We have $a(0) = 3 - 2\sqrt{2} \approx .1716$. It seems interesting to determine the largest value of $a(\delta)$ for which the Lemma is true, at least when $\delta = 0$. It is plausible that the extreme functions when $\delta = 0$ are

$$u_1(z) = \Re \frac{(1+z)(z^2 - 4z + 1)}{(1-z)(1+z^2)} \quad \text{and} \quad u_2(z) = u_1(-z).$$

This example shows that $a(0) \leq 2 - \sqrt{3} \approx .268$.

Proof of Lemma 1. It is enough to consider the case when $r = 1$ and $z_0 = 0$.

We always denote by $\vee$ and $\wedge$ the pointwise maximum and minimum of functions respectively. When $|z| = 1$ we have $u_+(z) = (u_1 \vee u_2)(z)$ and $u_-(z) = (u_1 \wedge u_2)(z)$. Thus

$$u_1 + u_2 = u_+ + u_-, \tag{3}$$

so the condition $u_-(0) + \delta u_+(0) \geq 0$ combined with (3) implies

$$u_1(0) + u_2(0) \geq (1-\delta) u_+(0).$$

It follows that one of the numbers $u_1(0)$, $u_2(0)$ is at least $(1-\delta)u_+(0)/2$. Suppose that

$$u_1(0) \geq \frac{1-\delta}{2} u_+(0). \tag{4}$$

Applying Harnack's inequality to the positive harmonic function u_+ we obtain

$$u_+(z) \geq \frac{1-r}{1+r} u_+(0), \quad |z| \leq r. \tag{5}$$

On the other hand, $u_+ - u_1$ is also a positive harmonic function, whose value at 0 is at most $(1+\delta)u_+(0)/2$, in view of (4). Thus Harnack's inequality implies

$$(u_+ - u_1)(z) \leq \frac{(1+\delta)(1+r)}{2(1-r)} u_+(0), \quad |z| \leq r. \tag{6}$$

Combining (5) and (6), we obtain for $|z| \le r$

$$u_1(z) \ge \left(\frac{1-r}{1+r} - \frac{(1+\delta)(1+r)}{2(1-r)}\right) u_+(0) = \frac{2(1-r)^2 - (1+\delta)(1+r)^2}{2(1-r^2)} u_+(0).$$

The last expression is positive when $r < a(\delta)$, where $a(\delta)$ is given by (2).

3. Proof of the Modified Conjecture for $p = 5$.

In view of Cartan's theorem we may assume that $\{1, 3\}$ and $\{2, 4\}$ are C-classes (in the full unit disk). Furthermore we may assume that $f_5 = -1$. Thus we have

$$f_1 + f_2 + f_3 + f_4 = 1, \tag{7}$$

and by (ii) in the definition of a C-class

$$f_3/f_1 \to -1 \quad \text{and} \quad f_4/f_2 \to -1, \quad \text{as} \quad f \to \mathcal{F}, \tag{8}$$

uniformly on compacta in $|z| < 1$. Our goal is to show that either f_5/f_1 or f_5/f_2 tends to zero uniformly on compacta in $|z| < r^* = 2^{-8}$; that is the index 5 can be added to one of the C-classes which already exist. In other words we want to show that one of the functions f_1 or f_2 tends to infinity uniformly on compacta in $|z| < r^*$.

Set $g_1 = f_1 + f_3,\ g_2 = f_2 + f_4$ and $g = g_1'$ (derivative), so that by (7)

$$g_1 + g_2 = 1 \tag{9}$$

and

$$g_1' = -g_2' = g. \tag{10}$$

We conclude from (8) that

$$f_1/g_1 = (1 + f_3/f_1)^{-1} \to \infty, \quad f \to \mathcal{F} \tag{11}$$

and similarly

$$f_2/g_2 \to \infty, \quad f \to \mathcal{F} \tag{12}$$

uniformly on compacta in $|z| < 1$.

Now it follows from (9) that

$$\log^+ |g_1| = \log^+ |1 - g_2| \le \log^+ |g_2| + \log 2$$

and similarly $\log^+|g_2| \leq \log^+|g_1| + \log 2$. Thus

$$\left|\log^+|g_1| - \log^+|g_2|\right| \leq \log 2. \tag{13}$$

Again from (9) we conclude that

$$|g_1| \vee |g_2| \geq 1/2, \tag{14}$$

so we may assume without loss of generality that

$$|g_1(0)| \geq 1/2. \tag{15}$$

Now we put $r^* = 2^{-8}$ and consider three cases.

Case 1.

$$|g_1(z)| \leq 2e^e \quad \text{for} \quad |z| \leq r^*. \tag{16}$$

We apply Cartan's lemma [6, Ch. VIII, §3] to estimate $|g_1|$ from below, using (15) and (16). For any given $\epsilon > 0$ we have

$$|g_1(z)| \geq C(\epsilon) \quad \text{for} \quad |z| = t$$

with some $t \in [r^* - \epsilon, r^*]$. So $|f_1(z)| \to \infty$ when $|z| = t$ in view of (11), and hence, by the Minimum Principle, $f_1(z) \to \infty$ uniformly in $|z| \leq r^* - \epsilon$.

Case 2. Now we assume that

$$|g_1(z_0)| \geq 2e^e \quad \text{for some} \quad z_0,\ |z_0| \leq r^*,$$

but $|g(z)| \leq 1$ for all z in the disk $|z| \leq r^*$.

Then we integrate

$$g_1(z) = g_1(z_0) + \int_{z_0}^{z} g(\zeta)\, d\zeta$$

and obtain $|g_1(z)| \geq 1,\ |z| \leq r^*$. Again (11) concludes the proof in this case.

Case 3. It remains to consider the possibility that there are points z_0 and z_1 in the disk $|z| \leq r^*$ such that

$$|g_1(z_0)| > 2e^e \tag{17}$$

and

$$|g(z_1)| \geq 1. \tag{18}$$

In view of (13) and (17) we have

$$|g_2(z_0)| \geq e^e. \tag{19}$$

Inequalities (11) and (17) imply

$$f_1(z_0) \to \infty. \tag{20}$$

For each $f \in F$ we fix reference points z_0 and z_1 in $D(r^*)$ satisfying (17) and (18)

Our plan is the following. We are going to apply Lemma 1 to the harmonic functions $u_1 = \log|f_1|$ and $u_2 = \log|f_2|$ in an appropriately chosen disk $D(z_0, r)$, with $r > 1/2$. The least harmonic majorant of $u_1 \vee u_2$ is positive by (11), (12) and (14). We need an estimate for the greatest harmonic minorant u_- of the function $u_1 \wedge u_2$ at the point z_0 from below. This is the same as the average of u_- over the circle $|z - z_0| = r$. To estimate this average from below we will use the derivative g and the subharmonic function $w = \log|g|$. We will show that (up to a small error term) w is a *subharmonic* minorant for $\log|g_1| \wedge \log|g_2| < u_1 \wedge u_2$, and thus $w(z_0)$ is a minorant for $u_-(z_0)$. However instead of a lower estimate of w at z_0 we only have an estimate at a nearby point z_1 (see (18)). We will handle this with the help of Lemma 3. Now we go into details.

For a holomorphic function h in the unit disk and positive number $r < 1 - r^*$ we define

$$m_{z_0}(r, h) = \int_{-\pi}^{\pi} \log^+ |h(z_0 + re^{i\theta})| \frac{d\theta}{2\pi}, \quad |z_0| < r^*.$$

Since $\log^+|h|$ is subharmonic, $m_{z_0}(r, h)$ increases with r. We will omit the index z_0 in this notation with understanding that the point z_0 specified above is always used. In what follows we use the notation C_k for absolute constants (they may be different in each occurence). We need the Lemma on the Logarithmic Derivative. It is convenient to start with the formulation as in [5, Sect. 2.2.2]: *for holomorphic functions g_i we have for* $1/2 < r < R < 1 - r^*$

$$m(r, g/g_i) \leq C_1 + C_2 \log m(R, g_i) + C_3 \log \frac{1}{R - r} + C_4 \log^+ \log^+ \frac{1}{|g_i(z_0)|}.$$

In view of (17) and (19) the last term can be omitted. We also need to eliminate the term with $\log(R-r)$. This can be done with the following lemma which goes back to E. Borel (see, for example [6, Ch. VIII, Lemma 1.4]).

Lemma 2 *Let $S \geq 0$ be an increasing function on $[0,b]$, $b>0$ and $\gamma>0$. Then there exists a subset $E \subset [0,b]$ of measure at most $2e^{-S(0)/\gamma}$, such that*

$$S(r+e^{-S(r)/\gamma}) \leq S(r)+\gamma \log 2, \quad r \notin E.$$

We choose $S(r)=\log m(r,g_i)$ (so that $S(0) \geq 1$ by (17) and (19)), $\gamma = S(0)/(3\log 2)$ and put $R = r+e^{-S(r)/\gamma}$ in the Lemma on Logarithmic Derivative. The exceptional set E in Lemma 2 has measure at most $1/4$, and the Lemma on Logarithmic Derivative becomes: *there exists* r,

$$1/2 = 2^7 r^* < r < 1-r^* \tag{21}$$

such that

$$m(r, g/g_i) \leq C_1 + C_2 \log m(r,g_i), \quad i=1,2; \tag{22}$$

where C_1 and C_2 are absolute constants. We fix this r satisfying (21) and (22) until the end of the proof. (Of course r, as well as z_0 and z_1, depends on f.)

Denote by u_+ the least harmonic majorant of $\log|f_1| \vee \log|f_2|$ in the disk $|z-z_0|<r$. Then by (11), (12) and (14)

$$u_+ \to +\infty, \quad f \to \mathcal{F}. \tag{23}$$

uniformly in $D(z_0,r)$. Using (11), (12) positivity and harmonicity of u_+ we obtain

$$m(r,g_i) \leq m(r,f_i) \leq \int_{-\pi}^{\pi} u_+(z_0+re^{i\theta})\frac{d\theta}{2\pi} = u_+(z_0), \quad i=1,2. \tag{24}$$

Thus (22), (24) and (23) imply for $i=1,2$

$$m(r,g/g_i) \leq C_1 + C_2 \log m(r,g_i) \leq C_1 + C_2 \log u_+(z_0) \leq o(u_+(z_0)) \tag{25}$$

as $f \to \mathcal{F}$. It follows from (24) and (25) that

$$m(r,g) \leq m(r,g_1) + m(r,g/g_1) \leq (1+o(1))u_+(z_0), \quad f \to \mathcal{F}. \tag{26}$$

Now we need the following

Lemma 3 *Let g be an analytic function in the disk $|z - z_0| \leq r$ and suppose that $|g(z_1)| \geq 1$ for some $z_1 \in D(z_0, r)$. Then*

$$\int_{-\pi}^{\pi} \log|g(z_0 + re^{i\theta})| \frac{d\theta}{2\pi} + \delta m(r, g) \geq 0,$$

where $\delta = 4r|z_0 - z_1|/(r - |z_0 - z_1|)^2$.

Proof. Assume without loss of generality that $z_0 = 0$ and put $|z_1| = t$. Then by Poisson's formula

$$\begin{aligned} 0 &\leq \log|g(z_1)| \\ &\leq \frac{r+t}{r-t} \int_{-\pi}^{\pi} \log^+|g(re^{i\theta})| \frac{d\theta}{2\pi} - \frac{r-t}{r+t} \int_{-\pi}^{\pi} \log^-|g(re^{i\theta})| \frac{d\theta}{2\pi} \\ &= \frac{r-t}{r+t} \int_{-\pi}^{\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi} + \frac{4rt}{r^2 - t^2} m(r, g). \end{aligned}$$

This proves the lemma.

In our situation we have $|z_0 - z_1| \leq 2r^* = 2^{-7}$, so by (21) the number δ from Lemma 3 has the following bound:

$$\delta \leq \frac{8rr^*}{(r - 2r^*)^2} \leq \frac{16r^*}{r} \leq 2^{-3}. \tag{27}$$

We apply Lemma 3 to our function g and use (18) (27) and (26) to obtain

$$\int_{-\pi}^{\pi} \log|g(z_0 + re^{i\theta})| \frac{d\theta}{2\pi} + \left(\frac{1}{8} + o(1)\right) u_+(z_0) \geq 0, \quad f \to \mathcal{F}. \tag{28}$$

Finally we estimate $|g|$ from above:

$$\log|g| \leq \log|g_i| + \log^+|g/g_i|, \quad i = 1, 2.$$

These inequalities together with (11) and (12) imply

$$\begin{aligned} \log|g| &\leq \log|g_1| \wedge \log|g_2| + \log^+|g/g_1| + \log^+|g/g_2| \\ &\leq \log|f_1| \wedge \log|f_2| + \log^+|g/g_1| + \log^+|g/g_2|. \end{aligned}$$

We integrate this inequality over the circle $|z - z_0| = r$ and use (25) to estimate the integrals involving logarithmic derivatives:

$$\int_{-\pi}^{\pi} (\log|f_1| \wedge \log|f_2|)(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \geq \int_{-\pi}^{\pi} \log|g(z_0 + re^{i\theta})| \frac{d\theta}{2\pi} + o(u_+(0)),$$

which with (28) gives

$$\int_{-\pi}^{\pi} (\log|f_1| \wedge \log|f_2|)(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} + \left(\frac{1}{8} + o(1)\right) u_+(z_0) \geq 0, \quad f \to \mathcal{F},$$

so we are in position to apply Lemma 1 with $\delta < 1/8$. Using this Lemma we conclude that either $\log|f_1|$ or $\log|f_2|$ tends to infinity in the disk

$$D(z_0, a'r), \tag{29}$$

where $a' < a(1/8)$ and $r > 1/2$ (see (21)). A simple computation with (2) shows that $a(1/8) > 1/7$, so we may take $a' = 1/10$ and then $a'r > 1/20$. We also have $|z_0| \leq r^* = 2^{-8}$, thus the disk (29) contains $D(0, r^*)$ and this finishes the proof.

4. A counterexample to Cartan's conjecture. For $|z| < 1$ and positive integer n put

$$g_{1,n} = \sqrt{n} \int_{-1}^{z} e^{-n\zeta^2} d\zeta$$

and

$$g_{2,n} = g_{1,n}(-z) = \sqrt{n} \int_{z}^{1} e^{-n\zeta^2} d\zeta,$$

so that

$$g_{1,n} + g_{2,n} = \sqrt{n} \int_{-1}^{1} e^{-n\zeta^2} d\zeta = c_n = \sqrt{\pi} + o(1), \quad n \to \infty. \tag{30}$$

Elementary estimates show that

$$|g_{i,n}(z)| \leq 2\sqrt{n} e^n, \quad |z| < 1, \quad i = 1, 2$$

and

$$|g_{1,n}(z)| \leq \sqrt{n} e^{-n/2}, \quad \Re z < -\frac{\sqrt{3}}{2}, \ |z| < 1.$$

Thus if we put

$$f_{1,n}(z) = \exp\{n(14(z+1) - 1/3)\}$$

and $f_{2,n}(z) = f_{1,n}(-z)$ then

$$g_{1,n}(z) = o(f_{1,n}(z)), \quad n \to \infty, \tag{31}$$

uniformly in $D(1)$, and

$$g_{2,n}(z) = o(f_{2,n}(z)), \quad n \to \infty, \tag{32}$$

uniformly in $D(1)$.

Evidently $f_{1,n}$ and $f_{2,n}$ are units. So are $f_{3,n} := -f_{1,n} + g_{1,n}$ and $f_{4,n} := -f_{2,n} + g_{2,n}$ in view of (31) and (32). If we put $f_{5,n} := -c_n$ then it is also a unit (just a constant) and

$$f_{1,n} + f_{2,n} + f_{3,n} + f_{4,n} + f_{5,n} = 0$$

in view of (30).

It remains to notice that $f_{5,n}$ cannot belong to any C-class. Indeed, none of the sequences $f_{i,n},\ 1 \leq i \leq 4$ is bounded from above or away from zero on compacta in $D(1)$. Thus by (30) none of the quotients $f_{i,n}/f_{5,n}$ can be normal in $D(1)$.

Addition of April 24, 1996. P. M. Tamrazov constructed an example which shows that the expression (2) gives the largest value of $a(\delta)$ for which the statement of Lemma 1 is true, for every $\delta \in (0,1)$. Thus Lemma 1 gives the best possible estimate and the conjecture stated in the Remark after Lemma 1 is wrong.

We describe the example with P. M. Tamrazov's permission. Let

$$P(z,t) = \Re \frac{e^{it} + z}{e^{it} - z}$$

be the Poisson kernel. Put

$$u_\epsilon = P(\,.\,,\pi) - \frac{1+\delta}{4}\left(P(\,.\,,\epsilon) + P(\,.\,,-\epsilon)\right).$$

A straightforward computation shows that u_ϵ has a positive zero which tends to $a(\delta)$ as $\epsilon \to 0$, where $a(\delta)$ is given by (2). On the other hand, if ϵ_1 and ϵ_2

are two different numbers on $(0,\pi)$ then u_{ϵ_1} and u_{ϵ_2} satisfy all conditions of Lemma 1 because in this case $u_+ = P(.\,,\pi) > 0$ and

$$u_-(0) = P(0,\pi) - \frac{(1+\delta)}{4}\left(P(0,\epsilon_1) + P(0,-\epsilon_1) + P(0,\epsilon_2) + P(0,-\epsilon_2)\right) = -\delta,$$

so $\delta u_+(0) + u_-(0) = 0$.

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