Complex numbers

January 27, 2021

Complex numbers are expressions of the form $z = x + iy$ where x, y are real numbers, and $i^2 = -1$ (by definition). Complex numbers can be added by the rule

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

so we can associate to a complex number a vector (x, y) in the plane $\mathbf{R}^2$ and the addition rule is the same as for vectors. Similarly you can multiply complex numbers by real numbers, and obtain a real vector space of dimension 2.

The real numbers x and y are called the real and imaginary parts of the complex number $z = x + iy$; they are denoted by $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$. Numbers of the form $x + i \cdot 0$ are identified with real numbers x. Numbers of the form $iy = 0 + iy$ are called pure imaginary.

Multiplication of complex numbers is defined like multiplication of polynomials of degree 1 in the variable i, but i^2 is replaced everywhere by -1, so higher powers of i never occur:

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_2y_2) + i(x_1y_2 + x_2y_1).$$

With these rules of addition and multiplication, complex numbers form a *field*, that is a collection of objects with two operations on them (addition and multiplication) which obey all usual rules (commutative, associative and distributive laws, and division is possible on every complex number except $0 = 0 + i0$.

To perform division, we write:

$$\frac{1}{x + iy} = \frac{x - iy}{(x - iy)(x + iy)} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

The RHS is an expression of the form $a + ib$ with real a, b, that is a complex number. The only exception is $x = y = 0$.

For a complex number $z = x + iy$ the number $\overline{z} = x - iy$ is called the *conjugate*. Operation of conjugation respects addition and multiplication:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}.$$

This implies that whenever we have a correct formula with complex numbers, conjugating all terms in this formula gives a correct formula.

The real and imaginary parts of z can be expressed in terms of conjugation operation

$$\mathrm{Re}\, z = (z + \overline{z})/2, \quad \text{and} \quad \mathrm{Im}\, z = (z - \overline{z})/(2i). \tag{1}$$

If $z = x + iy$ is a complex number then

$$z\overline{z} = x^2 + y^2 \geq 0$$

is a non-negative real number, so there exists a non-negative square root of it which is called the absolute value and denoted by $|z| = \sqrt{z\overline{z}}$. This is the same as the length of a vector if complex numbers are interpreted as vectors in $\mathbf{R}^2$. The absolute value of a product is the product of absolute values: $|z_1 z_2| = |z_1||z_2|$.

We have the following inequalities:

$$|\mathrm{Re}\, z| \leq |z|, \quad |\mathrm{Im}\, z| \leq |z|, \quad |z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|. \tag{2}$$

It follows that

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\,(z_1\overline{z_2}) \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ &= (|z_1| + |z_2|)^2. \end{aligned}$$

Taking square roots we obtain

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This is called the triangle inequality. Replacing $z_2 \mapsto z_2 - z_1$ we obtain $|z_2 - z_1| \geq |z_2| - |z_1|$, but z_1 and z_2 can be interchanged, so

$$|z_1 - z_2| \geq \left||z_1| - |z_2|\right|.$$

The notion of absolute value allows us to define a *distance* between two complex numbers as $|z_1 - z_2|$, and the notion of distance permits to consider limits. The definition of the limit is the same as for real numbers: we say that $\lim z_n = a$ if for every $\epsilon > 0$ there exists a positive integer N such that $|z_n - a| < \epsilon$ for all $n > N$.

In view of the inequalities (2), $\lim z_n = a$ if and only if $\lim \operatorname{Re} z = \operatorname{Re} a$ and $\lim \operatorname{Im} z = \operatorname{Im} a$. So convergence of a complex sequence is just equivalent to the convergence of real and imaginary parts.

Now we consider functions and equations with complex variables. Since the definition of a polynomial uses only addition and multiplication, it is the same as for real numbers. So we can consider quadratic equations.

Example. Solve $z^2 = i$. We write $z = x + iy$, then

$$z^2 = x^2 - y^2 + 2ixy.$$

This must be equal to i, so we obtain a system

$$x^2 - y^2 = 0, \quad 2xy = 1.$$

Eliminating y from the second equation and substituting to the first, we obtain $x^4 = 1/4$, which has two (real) solutions $y = \pm 1/\sqrt{2}$, then $x = \pm 1/\sqrt{2}$, and the second equation shows that x and y must be of the same sign. So we obtain two solutions $z_{1,2} = \pm(1 + i)/\sqrt{2}$.

Similarly one can show that equation $z^2 = w$ has exactly two solutions for every complex $w \neq 0$. When $w = 0$ there is one solution, $z = 0$.

Once we know how to solve this equation, we can solve any quadratic equation by the quadratic formula, which is the same as in the algebra of real numbers. Unlike in real algebra, every complex number has a square root, so every quadratic equation has one or two complex solutions.

It is a remarkable fact that *every* polynomial equation has at least one complex solution. This is called the *Fundamental theorem of algebra.*

If P is a non-zero polynomial, and $P(z_1) = 0$, then P factors: $P(z) = (z - z_1)Q(z)$, for some polynomial Q. This is obtained by the procedure of "long division" (with remainder) of polynomials which is similar to division of integers. When we multiply polynomials, their degrees are added, so $\deg P = \deg Q + 1$. So from the fundamental theorem of algebra we obtain that *Every polynomial of degree $d \geq 1$ factors into polynomials of degree 1:*

$$P(z) = c(z - z_1)(z - z_2) \ldots (z - z_d),$$

where $c, z_1, \ldots, z_d$ are complex numbers.

These numbers $z_1, \ldots, z_d$ are called the roots of the polynomial. They do not have to be distinct. Grouping the same roots, we obtain another form of factorization

$$P(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_k)^{m_k}$$

where $m_1 + \ldots + m_k = d = \deg P$ and this time $z_1, \ldots, z_k$ are distinct. The numbers m_j are called *multiplicities* of the roots z_j.
Example. Consider a polynomial equation with *real* coefficients $P(z) = 0$. If $P(z_1) = 0$ then we can apply complex conjugation, and obtain $P(\overline{z_1}) = 0$. So non-real roots must come in pairs of complex conjugate roots. As a result, every real polynomial of degree at least 1 factors into real polynomials of degrees 1 and 2.

Among other functions of complex variable, the most important is the exponential function. It is defined as a sum of the infinite series

$$e^z = 1 + z + z^2/2 + z^3/6 + \ldots = \sum_0^\infty \frac{z^n}{n!}.$$

So it is a limit of polynomials (truncated sums), and one can show that this series in convergent for all complex values of z. Such functions (defined as limits of polynomials which exist for all complex numbers) are called *entire functions.*

The definition coincides with the Taylor series of the exponential function, so for real z our function coincides with the real exponential function. Convergent power series can be differentiated term-by-term, using the differentiation rule $(d/dz)z^n = nz^{n-1}$, and we obtain the first two main properties of the exponential:

$$e^0 = 1, \quad \text{and} \quad (d/dz)e^z = e^z.$$

The third main property is that the exponential function transforms addition to multiplication:

$$e^{z_1+z_2} = e^{z_1}e^{z_2}. \tag{3}$$

To prove it, we use the binomial formula.

$$e^{z_1+z_2} = \sum_{n=0}^\infty \frac{(z_1+z_2)^n}{n!} = \sum_{n=0}^\infty \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{m+k=n} \frac{z_1^k}{k!} \frac{z_2^m}{m!} \quad \text{(we changed} \quad k \quad \text{to} \quad m = n-k)$$
$$= \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{m=0}^{\infty} \frac{z_2^m}{m!} = e^{z_1} e^{z_2}.$$

Applying this to $z = x + iy$ we obtain $e^{x+iy} = e^x e^{iy}$. To understand what e^{iy} is, we take real and imaginary part. Notice that i^{2m} is real and equals $(-1)^m$, while i^{2m+1} is pure imaginary and equals $i(-1)^m$. So we have, separating even and odd n in the series of the exponential:

$$e^{iy} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} y^{2m} + i \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} y^{2m+1}.$$

You can recognize in the real and imaginary parts of e^{iy} the Taylor series of cos and sin. So we have the **fundamental formula**

$$e^{x+iy} = e^x (\cos y + i \sin y). \tag{4}$$

Notice that $e^z e^{-z} = 1$ for all z; this follows from (3). When y is real we also have $e^{-iy} = \overline{e^{iy}}$. Using (1) we obtain $\cos(y) = (e^{iy} + e^{-iy})/2$ and $\sin y = (e^{iy} - e^{-iy})/(2i)$. This suggests the definition for all complex z:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} z^{2m},$$

and

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} z^{2m+1},$$

so cos and sin are also entire functions. Notice that all trigonometric formulas can be easily derived from these definitions and the three fundamental properties of the complex exponential. One also defines hyperbolic functions

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cos(iz),$$

and

$$\sinh z = \frac{e^z - e^{-z}}{2} = -i \sin(iz).$$

This is the minimum information about complex numbers you need for this course. More is contained in Appendix 2 of the book.