2. Complex and real vector spaces. In the definition of a vector space there is a set of numbers (scalars) which can be an arbitrary field. Thus we have real and complex vector spaces. If V is a complex vector space, we can consider only multiplication of vectors by real numbers, thus obtaining a real vector space, which is denoted $V_{\mathbf{R}}$.

2.1 If $\{\mathbf{a}_1 \ldots, \mathbf{a_n}\}$ is a basis in V then

$$\{\mathbf{a}_1, i\mathbf{a}_1, \ldots, \mathbf{a_n}, i\mathbf{a}_n\} \tag{1}$$

is a basis in $V_{\mathbf{R}}$, so the dimension of $V_{\mathbf{R}}$ is $2n$. They usually express this as $\dim_{\mathbf{R}} V = 2n$, keeping the same symbol for V and $V_{\mathbf{R}}$.

Now suppose that a real vector space U is given. How to introduce on it a structure of a complex vector space, so that addition and multiplication by real numbers remain unchanged? We have to introduce multiplication of vectors in U by complex numbers. Let us consider multiplication by i first of all. It is clear that $J(\mathbf{a}) = i\mathbf{a}$ is a linear operator in U (verify!), and this linear operator has the property $J^2 = -I$, where I is the unit operator. In the opposite direction, we have

2.2 Let U be a real vector space, and J a linear operator on U with the property $J^2 = -I$. Then we can define multiplication by complex numbers by $(x+iy)\mathbf{a} = x\mathbf{a} + yJ(\mathbf{a})$, and U with such multiplication will be a complex vector space.
2.3 Linear operators with the property $J^2 = -I$ in a real vector space exist iff the dimension is even. For each such operator there is a basis where its matrix has the form

$$\begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{pmatrix}.$$

Such an operator is called *a complex structure* on a real vector space U.

Suppose now that two complex vector spaces V and V' are given. We denote by J and J' the operators of multiplication by i in these spaces. Let

A be a *real* linear map between these space. (They say $A \in \mathrm{Hom}_{\mathbf{R}}(V_1, V_2)$, instead of $A \in \mathrm{Hom}(V_{\mathbf{R}}, V'_{\mathbf{R}})$). Then A is a *complex linear map* iff $J'A = AJ$, that is A "commutes with the multiplication by i". If $V' = V$, $\mathrm{d}im_{\mathbf{R}} V = 2$, and a basis is chosen as in (1), then the matrix (a_{ij}) of A in this basis satisfies

$$a_{11} = a_{22} \quad \text{and} \quad a_{12} = -a_{2,1}. \tag{2}$$

Compare this with 1.3. What does this mean geometrically?

If V is a real or complex vector space, a real-valued function $|.|$ on V is called a *norm* if it satisfies a), b), c) in 1.5. (There is no multiplication of two vectors in a vector space, so in b) we mean that z_1 is a scalar, and $|z_1|$ in the RHS stands for the norm in the scalar field). In the usual way (as in 1.5–1.6) a norm defines a distance, and a distance defines a topology. If a normed vector space is complete, it is called a *Banach space.* Banach spaces can be real or complex. Banach spaces constitute a natural setting for "multivariate calculus". A good calculus textbook, written on this level of generality is H. Cartan, *Differential Calculus* (Translated from French), Paris, Hermann, and Boston, Haughman, 1971.

The following are examples of norms in $\mathbf{C}_n$ or $\mathbf{R}^n$:

2.4 The *sup*-norm: $|z|_\infty = \sup\{|z_1|, \ldots, |z_n|\}$ and
the L^p-norm: $|z|_p = (|z_1|^p + \ldots + |z_n|)^{1/p}, \; p \geq 1$.

A very important case is when $p = 2$. The-L^2 norm is associated with the standard Hermitian product, which is defined in $\mathbf{C}^n$ by

$$(z, w) = z_1 \bar{w}_1 + \ldots z_n \bar{w}_n.$$

(Compare 1.8). We have $|z|_2^2 = (z, z)$.

2.5 The standard Hermitian product has the following properties:

a) $(z, z) \geq 0$ with equality iff $z = 0$,
b) $(z, w) = \overline{(w, z)}$,
c) $(az_1 + bz_2, w) = a(z_1, w) + b(z_2, w)$.

Here z and w are arbitrary vectors, and a, b are complex numbers. Any function $V \times V \to \mathbf{C}$ with these properties on a complex vector space V is called an Hermitian product, and V equipped with such product is called a Hermitian space. A complete Hermitian space is called a *Hilbert space.*