3. Formal power series are just sequences of complex numbers, with operations of addition and multipllication defined in the following way. If $F = (a_n)$ and $G = (b_n)$ then $F + G = (a_n + b_n)$ and $FG = (c_n)$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 = \sum_{k=0}^{n} a_n b_{n-k}. \tag{1}$$

3.1 Power series with these operations form a commutative ring without divisors of zero. (This ring is actually a **C**-algebra, as it contains a copy of **C**).

It is a custom to write a power series $F = (a_n)$ as an infinite sum

$$F = a_0 + a_1 z + a_2 z^2 + \ldots = \sum_{n=0}^{\infty} a_n z^n, \tag{2}$$

which is a good mnemonics for the rules of addition and multiplication (1): they are analogous to those for polynomials. The standard notation for the ring of formal power series "of letter z" is $\mathbf{C}[[z]]$. Sometimes a power series (2) is called a *generating function* of the sequence (a_n), and the numbers $a_0, a_1 \ldots$ are called *coefficients* of F.

We stress that the signs $+$ and Σ have in general no meaning for infinite sums. Neither the symbols z^n have any meaning in (2), this is just a bookkeeping devise for application of addition and multiplication rules. In particular these z^n permit us to omit those members of a sequence which are equal to 0, so we can write $0 + 1z + 2z^2 + 0z^3 + 0z^4 + \ldots$ simply as $z + 2z^2$.

3.2 Verify the identities for power series:

$$(1 - z)(1 + z + z^2 + z^3 + \ldots) = 1,$$

$$(1 - z)^{-2}(1 + 2z + 3z^2 + 4z^3 + \ldots) = 1,$$

Find the product $(1 + z + z^2 + z^3 + \ldots)(1 + 2z + 3z^2 + 4z^3 + \ldots)$.
3.3 A power series has multiplicative inverse if and only if $a_0 \neq 0$.
3.4 Let a_n be the number of ways to pay n cents using the US coins, less than \$1 each, and F is the generating function given by (2). Then

$$F(z) = (1 - z)^{-1}(1 - z^5)^{-1}(1 - z^{10})^{-1}(1 - z^{25})^{-1}(1 - z^{50})^{-1}.$$

3.5* Find a_{100}.
3.6 The derivative of a power series $F = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$ is defined as the power series $dF/dz = a_1 + 2a_2 z + 3a_3 z^2 + \dots$. Show that the usual properties hold: $d(F+G)/dz = dF/dz + dG/dz$ and $d(FG)/dz = (dF/dz)G + F(dG/dz)$.

If we denote by $F(0)$ the "constant term" a_0 of a series (2), then coefficients can be expressed as

$$a_n = \frac{1}{n} F^{(n)}(0).$$

3.7 The *order* of a power series is defined as the index of the first (leftmost) non-zero coefficient. It is denoted by $\operatorname{ord} S$. Verify that the function $S \mapsto \exp(-\operatorname{ord} S)$ is a norm, that is satisfies a), b) and c) from 1.5. In fact, a stronger property that c) holds:
c') $|x+y| \leq \max\{|x|, |y|\}$.

Norms with such property are called for historical reasons *non-Archimedian*. (An *Archimedian* norm is one which *does not* have this property; all norms we encountered before were Archimedian).
3.8 Verify that $\mathbf{C}[[z]]$ is complete with respect to this norm, and that polynomials are dense in $\mathbf{C}[[z]]$. Describe in words, what convergence of a sequence means with respect to this norm.

Suppose that F and G are formal power series and $\operatorname{ord} G > 0$. Then one can *substitute* G into F: if F is given by (2) then

$$F \circ G = \sum_{n=0}^{\infty} a_n G^n.$$

Notice that $\operatorname{ord} G^n \geq n$, so the coefficients of $F \circ G$ are expressed as *finite* sums, which makes this definition possible.

3.9 This operation of composition is associative.
3.10 A formal series F has a compositional inverse $F^{[-1]}$ iff $F(0) = 0$ and $F'(0) \neq 0$. Thus power series with these two properties make a group, whose identity element is $F(z) = z$. This group is called the group of (formal) *parameters*; the reasons for this name will be seen later.

A formal *Laurent series* is an expression of the form

$$\sum_{n=m}^{\infty} a_n z^n, \tag{3}$$

where m is an integer, may be negative. The set of all Laurent series of the letter z is denoted by $\mathbf{C}((z))$. It contains $\mathbf{C}[[z]]$ and the operations of addition, multiplication and differentiation have natural extension from $\mathbf{C}[[z]]$ to $\mathbf{C}((z))$.

3.11 $\mathbf{C}((z))$ is a field. It is the quotient field of the ring $\mathbf{C}[[z]]$.

The order function ord has a natural extension to $\mathbf{C}((z))$, so that $\exp(-\operatorname{ord} F)$ still has all properties of a norm, thus $\mathbf{C}((z))$ is a normed field. One can substitute a formal local parameter into a formal Laurent series.

The coefficient a_{-1} in (3) plays a special role, so it has a special name, the *residue* of the series (3), and a notation $a_{-1} =: \operatorname{res} F$.

Theorem. *If F is a formal Laurent series, and G a formal local parameter, then*

$$\operatorname{res} F = \operatorname{res} \{(F \circ G)G'\}.$$

Proof. Evidently res is a linear functional, so it is enough to check the formula for the case $F(z) = z^m$. Suppose first that $m \neq -1$. Then

$$\operatorname{res} \{F \circ G)G'\} = \operatorname{res} G^m G' = \operatorname{res} \left(\frac{1}{m+1}(G^{m+1})'\right) = 0,$$

because the residue of a derivative is always zero. Now for $m = -1$ we denote coefficients of G by b_n and obtain

$$\operatorname{res} G^{-1}G' = \left((b_1)^{-1}z^{-1} + \ldots\right)(b_1 + \ldots) = 1.$$

This proves the theorem. □

As an application of this theorem, we can write an explicit formula for the coefficients of a compositional inverse of a power series.

Corollary (Bürmann–Lagrange Formula). *Let $F \in \mathbf{C}[[z]]$, and G a parameter. Then for $n > 0$, the n-th coefficient of $F \circ G^{[-1]}$ is equal to*

$$\frac{1}{n}\operatorname{res}(F'G^{-n}) = \frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}(F' \left\{\left(\frac{z}{G}\right)^n\right\}.$$

Proof. This coefficient is

$$\operatorname{res}\left(z^{-n-1}F\circ G^{-1}\right) = \operatorname{res}\left(G^{-n-1}FG'\right)$$

$$= -\frac{1}{n}\operatorname{res}\left(F(G^{-n})'\right) = \frac{1}{n}\operatorname{res}\left(F'G^{-n}\right).$$

3.12 As an example, find a closed formula for coefficients of the compositional inverse to the power series

$$G(z) = ze^{-z} = z - z^2 + \frac{z^3}{2!} - \ldots.$$