4. Summable sequences. A sequence $a = (a_n)$ of complex numbers (or more generally, elements of a Banach space) is called *summable* if

$$N = \sup \left\{ \sum_{n \in E} |a_n| : S \text{ is a finite set} \right\} < \infty.$$

4.1 All summable sequences form a vector space, and N is a norm in this vector space. This vector space is complete, and it is called ℓ_1.

For each summable sequence, the sequence of its partial sums (s_k),

$$s_k = \sum_{n=0}^{\infty} a_n, \quad k = 0, 1, 2 \ldots$$

is a Cauchy sequence, so it has a limit. This limit is called "the sum of the series"

$$\sum_{n=0}^{\infty} a_n. \tag{1}$$

Such series (whose terms form a summable sequence) are also called *absolutely convergent*.

4.2 Suppose that $n \mapsto m(n)$ is arbitrary permutation of integers (that is a bijection of the set of non-negative integers onto itself). If (1) is absolutely convergent, then

$$s^* = a_m(0) + a_m(1) + \ldots = \sum_{n=0}^{\infty} a_{m(n)}$$

is also absolutely convergent and has the same sum.

4.3 Sum is a linear functional of norm 1 on ℓ_1.

5. Uniform convergence. Let (f_n) be a sequence of functions, defined on some set E, with values in a Banach space. We say that this sequence is convergent to a function f *uniformly on* E if *for every* $\epsilon > 0$ *there exists an integer* N *such that for every* $x \in E$ *and every* $n \geq N$ *we have* $|f_n(x) - f(x)| < \epsilon$.

5.1 Uniform limit of continuous functiuons (defined on a topological space E) is continuous.

5.2 Suppose that the common domain of our functions is the segment $[0,1]$, and that they are continuous. If $f_n \to f$ uniformly then

$$\int f_n(x)dx \to \int f(x)dx, \quad n \to \infty.$$

5.3 Given a set $E \subset \mathbf{C}$ we consider the Banach space $C(E)$ of all continous complex-valued unctions on E with the sup-norm. Then the absolute convergence of a series in $C(E)$ implies uniform on E convergence of its partial sums. Absolute convergence of such functional series in $C(E)$ is usually called *normal* convergence on E.

Sometimes normally convergent series are called "uniformly and absolutely convergent", which is not precise:

5.4 Find an example of a series

$$\sum_{n=0}^{\infty} f_n \tag{2}$$

of continuous functions on $[0,1]$, such that for every $x \in [0,1]$ the series $\sum_{n=0}^{\infty} f_n(x)$ is absolutely convergent, and the sequence of partial sums of (2) is uniformly convergent on $[0,1]$, but (2) is not normally convergent.

6. Convergent power series. If $F \in \mathbf{C}[[z]]$, and z_0 is a complex number, we can substitute z_0^n for z^n and obtain a series of numbers, which may be absolutely convergent or not. Similarly, considering z^n as a function defined on some set $E \subset \mathbf{C}$, we obtain a series in the Banach space $C(E)$.

6.1 **Theorem**. *For every $F \in \mathbf{C}[[z]]$ there exists $R \in [0,+\infty]$ with the following properties:*
a) for each $r \in [0,R)$ the series is normally convergent on the set $\{z : |z| \leq r\}$, and
b) for each z such that $|z| > R$ the series $F(z)$ is divergent.

Proof. Let $R = \sup\{r \geq 0 : (|a_n|r^n|) \text{ is a bounded sequence}\}$. Then evidently b) holds. If $R = 0$ then a) is void. If $R > 0$, we take arbitrary $r \in (0,R)$ and put $r_1 = (R+r)/2$. Then for $|z| \leq r$ we have

$$|a_n z^n| \leq \left(\frac{r}{r_1}\right)^n |a_n| r_1^n \leq \text{const} \left(\frac{r}{r_1}\right)^n,$$

so the series is normally convergent for $|z| \le r$ because $r/r_1 \in (0,1)$. □

A power series F with $R > 0$ is called *convergent*. The sum of such series is a cointinuous function in $D(0,R)$.

6.2 The substitution map $\mathbf{C}[[z]] \to C(R)$ is a homomorphism of rings.

We are going to prove that this homeomorphism is injective. Actually we will prove a little stronger statement.

6.3 **Theorem**. *If $F \neq 0$ is a series whose radius of cponvergence is $R > 0$, then there exists $r \in (0,R)$, such that $F(z) \neq 0$ for $0 < |z| < r$.*

Proof. We have $m := \operatorname{ord} F < \infty$, so $F(z) = z^m G(z)$, where $G(0) \neq 0$. By continuity there exists an $r \in (0,R)$, such that $G(z) \neq 0$ for $|z| < r$. This proves the theorem. □

6.4 **Theorem**. *If F and G are convergent power series, and $G(0) = 0$, then $F \circ G$ is also convergent.*

Proof. Exercise!

6.5 **Corollary**. *If F is a convergent power series, and $F(0) \neq 0$ than the multiplicative inverse F^{-1} is also convergent.*