6. Regular functions We denote the set of all convergent power series by $\mathbf{C}_c[[z]]$. (The subscript c stands for "convergent".)

6.1 **Theorem**. *If $F \neq 0$ is a series whose radius of convergence is $R > 0$, then there exists $r \in (0, R)$, such that $F(z) \neq 0$ for $0 < |z| < r$.*

Proof. We have $m := \operatorname{ord} F < \infty$, so $F(z) = z^m G(z)$, where $G(0) \neq 0$. By continuity there exists an $r \in (0, R)$, such that $G(z) \neq 0$ for $|z| < r$. This proves the theorem.

6.2 **Theorem**. *If F and G are convergent power series, and $G(0) = 0$, then $F \circ G$ is also convergent.*

Proof. Exercise!

6.3 **Corollary**. *If F is a convergent power series, and $F(0) \neq 0$ than the multiplicative inverse F^{-1} is also convergent.*

Definition. *Let D be a region in $\mathbf{C}$. A function $f : D \to \mathbf{C}$ is called* regular *or* **holomorphic** *if for every $a \in D$ there exists a convergent power series $S \in \mathbf{C}[[w]]$ such that $f(z) = S(z - a)$ in a neighborhood of a.*

6.4 The set of all regular functions in a given region D is a ring (and a $\mathbf{C}$-algebra) without divisors of zero. This ring is denoted by $H(D)$

6.5 Polynomials are regular in $\mathbf{C}$.

6.6 Exponential function is regular in $\mathbf{C}$.

To obtain more examples we prove that the sum of a convergent power series is regular in its disc of convergence. This is not an immediate consequence from the definition!

Re-expansion Theorem. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series convergent in $|z| < R$. Then for every $a \in D(0, R)$ there exists a power series, convergent in $D(a, R - |a|)$, such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad z \in D(a, R - |a|).$$

Proof. Choose arbitrary $r \in (|a|, R)$. Then we have convergent series with positive terms

$$\infty > \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} (r - |a|)^k |a|^{n-k}.$$

Thus the double series

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a_n (z-a)^k a^{n-k}$$

is absolutely convergent, and its terms can be regrouped according to the powers of $(z-a)$:

$$f(z) = \sum_{k=0}^{\infty} (z-a)^k \left(\sum_{n=k}^{\infty} \binom{n}{k} a^{n-k} \right),$$

where both series (inside and outside the parentheses) are convergent.

Thus the sum of a convergent power series is a regular function in its disc of convergence. Now we prove the converse:

Theorem. *Let f be an regular function in a disc $D(a, R)$. Then the Taylor series of f at the point a has radius of convergence at least R.*

Proof. Suppose that the theorem is not true and the radius of convergence is $r \in (0, R)$. According to the definition of a regular function it can be expanded into a Taylor series at every point z of the closed disc $|z| \leq (R+r)/2$. Let $r(z)$ be the radius of convergence of this expansion. The re-expansion theorem implies that

$$r(z') \geq r(z) - |z - z'| \quad \text{and} \quad r(z) \geq r(z') - |z - z'|,$$

so $|r(z) - r(z') \leq |z - z'|$, thus $r(z)$ is a continuous function. as $r(z)$ is positive on the compact set $\{z : |z| < (R+r)/2\}$, it has a positive lower bound 2σ.

Let $M = \max_{|z| \leq r+\sigma} |f(z)|$. Then Cauchy's inequalities imply that

$$\frac{1}{n!} |f^{(n)}(z)| \leq M/\sigma^n, \quad |z| \leq r. \tag{1}$$

Differentiating the Taylor series

$$f(z)=\sum_{k=0}^{\infty} a_k z^k, \tag{2}$$

we obtain

$$\frac{1}{n!}|f^{(n)}(z)|=\sum_{n=k}^{\infty}\frac{k!}{n!(k-n)!}z^{k-n}.$$

Applying Cauchy's inequality to the last series, and using (1), we obtain

$$\frac{k!}{n!(k-n)!}|c_k||z|^{k-n}\leq M/\sigma^n, \quad |z|<r, n\leq k.$$

Passing to the limit, as $|z|\to r$, we can replace $|z|$ by r, so

$$\frac{k!}{n!(k-n)!}|c_k||r|^{k-n}\sigma^n\leq M.$$

adding these inequalities for $n=0,\ldots,k$ we obtain

$$|c_k|(r+\sigma)^k\leq M,$$

which implies that the radius of convergence of our series (2) is at least $r+\sigma$. This contradiction proves the theorem.

Cauchy's estimates and two theorems of this section constitute the core of the theory of analytic functions.

Now we draw many consequences from these results.

Definition. *A function f defined in a neighborhood of a point a is called* **C**-*differentiable, if the limit*

$$f'(a):=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}$$

exists.

6.7 If f is regular in a region D then f is **C**-differentiable at every point $a\in D$. The derivative f' is also a regular function in D.

Thus regular functions have **C**-derivatives of all orders, and these derivatives are also regular. Furthermore, if

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n \tag{3}$$

in a neighborhood of a point $a \in D$, then

$$a_n = \frac{f^{(n)}(a)}{n!},$$

That is the power series of f is actually its Taylor series.

Suppose that the radius of convergence of the series (3) is $R > 0$. Then the primitive in the sense of formal power series,

$$\sum_{n=1}^{\infty} \frac{a_{n-1}}{n} (z-a)^n$$

has the same radius of convergence. Thus a function regular in a disc has a primitive there, which is also regular.

6.8 The quotient field of $H(D)$ is a field. Its elements are called *meromorphic functions* in D. They do not map D into $\mathbf{C}$ but into $\overline{\mathbf{C}}$. A point $a \in D$ such that $f(a) = \infty$ is called a *pole* of f.

Every function, meromorphic in a neighborhood of a, which is not identically equal to zero, has a representation

$$f(z) = (z-a)^m g(z),$$

where m is an integer, positive or negative, and g is regular in a neighborhood of a, and $g(a) \neq 0$. The integer m is called the *order* of f at a, $m = \operatorname{ord}_a f$. If a is a pole then $m < 0$, and $-m$ is called *multiplicity of a pole* at a. If $f(a) = 0$, then $m > 0$, and we say that m is the multiplicity of zero at a in this case.