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On the Behavior of an Entire Function on a Sequence of Concentric Circles

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INTRODUCTION

This paper is an extended version of the work [4], [5]. Let f be an entire function of a finite order ρ, $f(0)=1$. For every $r>0$ consider the 2π-periodic function

$$g(\theta, r, f)=\frac{\log|f(re^{i\theta})|}{\log M(r,f)}, \qquad M(r,f)=\max_{|z|=r}|f(z)|.$$

It satisfies

$$\max_{\theta\in[0,2\pi]} g(\theta,r,f)=1, \qquad \int_0^{2\pi} g(\theta,r,f)\,d\theta \geqslant 0. \tag{1}$$

The second property follows from Jensen's inequality. If

$$\log M(r,f)=O(\log^2 r), \qquad r\to\infty, \tag{2}$$

then as W. Hayman proved [7]

$$\log|f(z)|\sim\log M(r,f), \qquad r=|z|, \tag{3}$$

as $z\to\infty$ outside a small exceptional set. This implies

$$g(\theta,r,f)\to 1 \quad \text{in} \quad L^1[0,2\pi], \quad r\to\infty. \tag{4}$$

Later J. Anderson showed that if

$$\log M(2r,f)\sim\log M(r,f), \qquad r\to\infty$$

then f satisfies (3), (4) (the exceptional set in this case can be larger than in Hayman's result) [1].

We shall show in Section 1 that for the functions f having slightly more rapid growth than (2) one can say nothing about $g(\theta,r,f)$ for all r except (1).

Denote by S_∞ the class of continuous 2π-periodic functions with the properties:

$$\max_{\theta\in[0,2\pi]} g(\theta) = 1, \qquad \int_0^{2\pi} g(\theta)\, d\theta \geq 0 \tag{5}$$

and let S_p be the closure of S_∞ in $L^p[0, 2\pi]$, $0 < p < \infty$.

THEOREM 1 *Let $\varphi(r)$ be a function tending to $+\infty$ as $r \to \infty$. Then there exists an entire function f, $f(0) = 1$, with the following properties*:

$$\log M(r, f) = O(\varphi(r) \log^2 r), \qquad r \to \infty; \tag{6}$$

and for arbitrary $g \in S_p$, $0 < p \leq \infty$ we have

$$\liminf_{r\to\infty} \|g(\theta, r, f) - g(\theta)\|_p = 0, \tag{7}$$

i.e. the closure of $\{g(\cdot, r, f)\}_{r\geq 1}$ in $L^p[0, 2\pi]$ is S_p. There exists also an entire function f with prescribed order ρ, $0 \leq \rho \leq \infty$ and satisfying (7).

Theorem 1 will be deduced from the following result which has an independent interest.

THEOREM 2 *Let a sequence of numbers $\eta_n \geq 1$ and a sequence of polynomials (P_n)*

$$P_n(0) = 1 \tag{8}$$

be given. Then there exists an entire function f with the property (6) *and for some increasing sequences $(T_n) \subset \mathbb{R}$, $(q_n) \subset \mathbb{N}$ both tending to ∞ we have*

$$\log|f(T_n z)| - q_n \log|P_n(z)| = o(q_n), \qquad n \to \infty \tag{9}$$

uniformly with respect to z, $1 \leq |z| \leq \eta_n$. There exists also an entire function f satisfying (9) *and having prescribed order.*

The proof of Theorem 2 is based on a construction from the paper [6].

Denote

$$m_p(r, f) = m_p(r, \infty, f) = \|\log^+|f(re^{i\theta})|\|_p,$$

$$m_p(r, 0, f) = m_p(r, f^{-1}), \qquad 0 < p \leq \infty.$$

Jensen's inequality is equivalent to

$$m_1(r, 0, f) \leq m_1(r, \infty, f).$$

In a conversation with one of the authors A. Ph. Grishin raised the following question: Do there exist a constant $C(p, \rho)$ such that every entire function f of the order ρ satisfies

$$m(r, 0, f) \leq C(p, \rho) m_p(r, \infty, f)?$$

(This question stimulated the present work.) The negative answer follows immediately from Theorem 1. Choosing a sequence $g_n \in S_\infty$ such that $\|g_{2n}^+\|_p = o(\|g_{2n}^-\|_p)$, $n \to \infty$, $p > 1$ and $\|g_{2n+1}^+\|_p = o(\|g_{2n+1}^-\|_p)$, $n \to \infty$, $p < 1$ we deduce the following

COROLLARY *For every ρ, $0 \leqslant \rho \leqslant \infty$ there exists an entire function f of the order ρ such that for every $p \neq 1$*

$$\limsup_{r\to\infty} \frac{m_p(r, 0, f)}{m_p(r, \infty, f)} = \infty. \tag{10}$$

If an entire function satisfies some regularity condition then the answer to Grishin's question is positive. We consider in Section 2 the entire functions satisfying

$$\log M(2r, f) = O(\log M(r, f)), \qquad r \to \infty, \tag{11}$$

which we rewrite as

$$\log M(er, f) \leqslant K \log M(r, f), \qquad r \geqslant r_0. \tag{12}$$

Such a class of entire functions may be characterized as having bounded orders of Pólya peaks [3].

THEOREM 3 *If an entire function f satisfies (12) then for every positive p and q we have*

$$m_q(r, 0, f) \leqslant C(p, q, K) m_p(r, \infty, f).$$

1. THE CONSTRUCTION OF "UNIVERSAL" ENTIRE FUNCTION

Proof of Theorem 2 We shall construct a function of the form

$$f(z) = \prod_{n=1}^{\infty} \{P_n(z/T_n)\}^{q_n}$$

and choose the sequences T_n, q_n tending to infinity to satisfy (6), (9).

We denote $x^\wedge = \max\{1, x\}$. Without loss of generality we may assume that $\varphi(r)$ is a continuously differentiable increasing function, $\varphi(r) \equiv 1$ for $r \in [0, e]$, $\varphi(r) \leqslant (\log r)^\wedge$ and $\varphi'(r) \leqslant 1/r$, $0 \leqslant r < \infty$ (if the last condition is not fulfilled then we replace $\varphi(r)$ for $r \geqslant e$ by the function

$$\max\left\{1, \int_1^r \min(\varphi'(t), 1/t)\, dt\right\} \quad).$$

It is clear that there exist positive constants A_k such that for all $r \geqslant 0$

$$\log M(r, P_k) \leqslant A_k \log(1 + r), \qquad k \in \mathbb{N}.$$

We construct inductively a sequence of positive numbers (T_n), $T_1 \geqslant e, T_{n+1} \geqslant \eta_n T_n$ a sequence of natural numbers (q_n) and a sequence of polynomials (Q_n) such that

$$\left|\log|Q_n(T_j z)| - q_j \log|P_j(z)|\right| \leqslant 2^{-j}(1 - 2^{-n}) q_j, \qquad 1 \leqslant |z| \leqslant \eta_j, \tag{14}$$

for all j, $1 \leqslant j \leqslant n$ and

$$\log M(r, Q_n) \leqslant (1 - 2^{-n}) \varphi(r) (\log^2 r)^\wedge, \tag{15}$$

for all $r > 0$.

Choose $T_1 \geqslant e$ so that

$$\log M(r/T_1, P_1) \leqslant \tfrac{1}{2}\varphi(r)(\log^2 r)^{\wedge}, \qquad r \geqslant 0. \tag{16}$$

Put

$$Q_1(z) = P_1(z/T_1), \qquad q_1 = 1. \tag{17}$$

The polynomial Q_1 has the properties (14)–(15).

We assume that $T_1, \ldots, T_{k-1}, q_1, \ldots, q_{k-1}$ and $Q_1, \ldots, Q_{k-1}$ have already been constructed satisfying (14) and (15). We show how to choose T_k, q_k, Q_k. Let $B_{k-1} > 1$ be such a number that for all $r > 0$

$$\log M(r, Q_{k-1}) \leqslant B_{k-1}(\log r)^{\wedge}. \tag{18}$$

Let r_k be large enough so that

$$(1 - 2^{-k})\varphi(r_k) \geqslant B_{k-1}(1 + 2^{k+4}A_k(\log \eta_k)^{\wedge}), \tag{19}$$

$$r_k \geqslant \eta_{k-1} T_{k-1}. \tag{20}$$

Now we choose T_k to satisfy the following conditions

$$T_k \geqslant r_k, \tag{21}$$

$$P_k(z) \neq 0, \qquad |z| \leqslant r_k/T_k, \tag{22}$$

$$2^{k+3}B_{k-1}\log(\eta_k T_k)|\log P_k(z/T_k)| \leqslant 2^{-2k}, \qquad |z| \leqslant r_k, \tag{23}$$

$$\big|\log|Q_{k-1}(T_k z)|\big| \leqslant 2B_{k-1}(\log(T_k|z|))^{\wedge}, \qquad |z| \geqslant 1. \tag{24}$$

(The branch of the logarithm $\log P_k(z/T_k)$ is chosen such that $\log P_k(0) = 0$ taking into account the fact that $P_k(z/T_k)$ has no zeros for $|z| \leqslant r_k$ by virtue of (22)). To show that we can satisfy (23) we note that for $|z| \leqslant r_k$ and for sufficiently large T

$$|\log P_k(T^{-1}z)| \log(T\eta_k) \leqslant 2|P_k(T^{-1}z) - 1| \log(T\eta_k)$$
$$\leqslant 2(1 + |P_k'(0)|)r_k T^{-1}\log(T\eta_k) = o(1), \qquad T \to \infty.$$

Now we put

$$q_k = [2^{k+3}B_{k-1}\log(\eta_k T_k)]$$

(here $[x]$ is the integral part of a number x),

$$Q_k(z) = Q_{k-1}\{P_k(z/T_k)\}^{q_k}.$$

From (20) and (21) it follows that $T_k \geqslant \eta_{k-1}T_{k-1}$. We show that (14) and (15) with $n = k$ are satisfied for Q_k. If $1 \leqslant |z| \leqslant \eta_j$, $1 \leqslant j \leqslant k-1$ then we have by virtue of (26), (14) (with $n = k-1$) and (25), (23), (20)

$$\big|\log|Q_k(T_j z)| - q_j \log|P_j(z)|\big| \leqslant \big|\log|Q_{k-1}(T_j z)| - q_j\log|P_j(z)|\big| + q_k\left|\log\left|P_k\left(\frac{T_j}{T_k}z\right)\right|\right|$$
$$\leqslant 2^{-j}(1 - 2^{-k+1})q_j + 2^{-2k} \leqslant 2^{-j}(1 - 2^{-k})q_j,$$

which gives (14) with $n = k$, $1 \leqslant j \leqslant n-1$.

By virtue of (26), (25), (24) we obtain

$$\left|\log|Q_k(T_k z)| - q_k \log|P_k(z)|\right| = \left|\log|Q_{k-1}(T_k z)|\right| \leqslant 2B_{k-1}\log(\eta_k T_k) \leqslant 2^{-k-1}q_k, \qquad 1 \leqslant |z| \leqslant \eta_k$$

which proves (14) with $n = k$, $j = k$.

Using (26), (18) and (13) we get for all $r \geqslant e$

$$\begin{aligned}\log M(r, Q_k) &\leqslant B_{k-1}\log r + q_k A_k \log\left(1 + \frac{r}{T_k}\right)\\ &\leqslant B_{k-1}\log r\left(1 + 2^{k+3}A_k \log(\eta_k T_k)\frac{\log(1 + r/T_k)}{\log r}\right)\\ &\leqslant B_{k-1}\log r\left(1 + 2^{k+3}A_k(\log \eta_k)^{\wedge}\log T_k\frac{\log(1 + r/T_k)}{\log r}\right). \qquad (27)\end{aligned}$$

Note that for $e \leqslant r \leqslant T_k$ we have

$$\log T_k \frac{\log(1 + r/T_k)}{\log r} \leqslant \frac{\log T_k}{T_k}\frac{r}{\log r} \leqslant 1 \leqslant 2\log r, \qquad (28)$$

and for $r \geqslant T_k$ we obtain

$$\log T_k \frac{\log(1 + r/T_k)}{\log r} \leqslant \log\left(1 + \frac{r}{T_k}\right) \leqslant 2\log r. \qquad (29)$$

It follows from (27)–(29) and (19) that for all $r \geqslant e$

$$\begin{aligned}\log M(r, Q_k) &\leqslant B_{k-1}\log r(1 + 2^{k+4}A_k(\log \eta_k)^{\wedge}\log r)\\ &\leqslant B_{k-1}(1 + 2^{k+4}A_k(\log \eta_k)^{\wedge})\log^2 r \leqslant (1 - 2^{-k})\varphi(r)\log^2 r.\end{aligned}$$

We see that (15) is satisfied for $n = k$.

This proves the possibility of construction of (q_n), (T_n) and (Q_n) with the properties (14) and (15).

We show that the sequence (Q_n) converges uniformly on compact subsets of the plane to some entire function f. By (26) the convergence of (Q_n) is equivalent to the convergence of the infinite product

$$\prod_{k=1}^{\infty}\left\{P_k\left(\frac{z}{T_k}\right)\right\}^{q_k}$$

and this product converges uniformly on compact sets in the plane by virtue of (23) and (25).

We show that the entire function f satisfies the requirements of Theorem 2. It is evident that $f(0) = 1$. Taking the limit in (14) for $n \to \infty$ we obtain (9). Taking the limit in (15) for $n \to \infty$ we get (6). Thus (6) is satisfied for f. This proves the first statement of Theorem 2.

To prove the second statement one has to change the above reasoning in the same manner as in [6, Theorem 2]. We omit the details.

Proof of Theorem 1 Let g be a continuous 2π-periodic function,

$$a = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta \geqslant 0. \tag{30}$$

We shall show that for every $\varepsilon > 0$ there exists a polynomial P such that

$$P(0) = 1, \tag{31}$$

$$\left|\log|P(e^{i\theta})| - g(\theta)\right| < \varepsilon, \qquad 0 \leqslant \theta \leqslant 2\pi. \tag{32}$$

Put

$$F(z) = \frac{z + e^{-a}}{1 + ze^{-a}} \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} g(\theta)\, d\theta\right\}.$$

Then

$$\log|F(e^{i\theta})| = g(\theta), \tag{33}$$

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})|\, d\theta - N(1, 0, F) = a - \int_{e^{-a}}^{1} \frac{dt}{t} = 0. \tag{34}$$

Thus we can define P as an appropriate Fejér's sum of F. Now (32) follows from (33) and (31) follows from (34).

Let us finish the proof of Theorem 1 using Theorem 2. Note that it is sufficient to prove (7) for a dense sequence $(g_n) \subset S_\infty$. Then (7) will be true for all $g \in S_\infty$ and consequently for all $g \in S_p$, $0 < p \leqslant \infty$. Without loss of generality we may assume that every function $g \in \{g_n\}$ belongs to the sequence infinitely often. Take a sequence $\varepsilon_n \to 0$ and construct the polynomials P_n satisfying (30) and

$$\left|\log|P_n(e^{i\theta})| - g_n(\theta)\right| < \varepsilon_n, \qquad 0 \leqslant \theta_n \leqslant 2\pi. \tag{35}$$

Using Theorem 2 with $\eta_n = 1$ we obtain an entire function f with the properties (6), (9). It follows from (5) that $\log M(1, P_n) \to 1$, $n \to \infty$ thus (7) is a consequence of (9), (35). Theorem 1 is proved.

2. ENTIRE FUNCTIONS OF A FINITE PÓLYA ORDER

We shall use the following lemmas.

LEMMA 1 *Let an entire function f satisfy* (12). *Put*

$$\theta(a, r) = \left|\{\theta: \log|f(re^{i\theta})| \geqslant a \log M(r, f)\}\right|, \qquad 0 < a < 1,$$

where $|E|$ denotes the length of a set $E \subset [0, 2\pi]$. Then

$$\operatorname{tg}\frac{\theta(a, r)}{4} \geqslant \frac{1}{\pi}\frac{1 - a}{K - 1}. \tag{36}$$

The proof is based on the reasoning of T. Carleman [2].

Proof Let $C(R) = \{z: |z| < R\}$, $M(R) = M(R, f)$,

$$D_R = \{z \in C(R): \log|f(z)| > a \log M(R)\}.$$

Denote by γ the part of the boundary of the set D_R which belongs to the circle $|z| = R$. Let $\omega(z, \gamma)$ be the harmonic measure of γ with respect to $C(R)$. Then we have for $z \in D_R$

$$\log|f(z)| \leqslant a \log M(R) + (1-a) \log M(R)\omega(z, \gamma). \tag{37}$$

Furthermore

$$\omega(z, \gamma) \leqslant h(r) = \frac{1}{2\pi} \int_{-\theta/2}^{\theta/2} \frac{R^2 - r^2}{R^2 - 2Rr\cos\psi + r^2}\, d\psi = \frac{2}{\pi}\, arctg\left(\frac{R+r}{R-r} \operatorname{tg}\frac{\theta}{4}\right), \tag{38}$$

where $\theta = \theta(a, R)$. Let ξ_r be a point with the properties $|\xi_r| = r$ and $\log|f(\xi_r)| = \log M(r)$. Take $\delta_0 > 0$ such that the poin $\xi_{R-\delta} = (R-\delta)e^{i\varphi(\delta)}$ belongs to D_R if $0 < \delta < \delta_0$. Using (37) with $z = \xi_{R-\delta}$ and (38) we obtain

$$\log M(R-\delta) - a \log M(R) - (1-a) \log M(R) h(R-\delta) \leqslant 0,$$

$$\frac{\log M(R) - \log M(R-\delta)}{\delta} \geqslant (1-a) \log M(R) \frac{1 - h(R-\delta)}{\delta}. \tag{39}$$

Let $\delta \to 0+$ in (39). Since $h(R) = 1$ we get

$$\frac{d}{dR} \log M(R) \geqslant (1-a) \log M(R) h'(R). \tag{40}$$

Since $h'(R) = \dfrac{1}{\pi R} \operatorname{ctg} \dfrac{\theta(a, R)}{4}$ it follows from (40) that

$$\frac{1-a}{\pi} \operatorname{ctg} \frac{\theta(a, R)}{4} \leqslant \frac{d}{dt} \log\log M(e^t), \tag{41}$$

where $t = \log R$.

To obtain (36) from (41) let us note that in view of convexity of $\log M(e^t)$ and (12) we have for $t = t_0 = \log r_0$

$$K - 1 \geqslant \frac{\log M(e^{t+1})}{\log M(e^t)} - 1 = \frac{1}{\log M(e^t)} \int_t^{t+1} \frac{d}{d\tau} \log M(e^\tau)\, d\tau \geqslant \frac{d}{dt} \log\log M(e^t).$$

Thus the lemma is proved.

LEMMA 2 *Let an entire function f satisfy* (12). *Then for all $p > 0$ we have*

$$\log M(r, f) \leqslant \left\{\frac{2(p+1)(1 + \pi^2(K-1)^2)}{K-1}\right\}^{1/p} m_p(r, f).$$

Proof We have

$$\frac{m_p(r, f)}{\log M(r, f)} \geqslant \left\{\frac{1}{2\pi} \int_0^1 \theta(a, r)\, d(a^p)\right\}^{1/p}.$$

Use Lemma 1 and integrate by parts. We get

$$\frac{m_p(r,f)}{\log M(r,f)} \geqslant \left\{-\frac{1}{2\pi}\int_0^{2\pi} a^p \frac{d}{da}\operatorname{arctg}\left(\frac{1}{\pi}\frac{1-a}{K-1}\right)da\right\}^{1/p}$$

$$= \left\{\tfrac{1}{2}(K-1)\int_0^1 \frac{a^p\,da}{(1-a)^2+\pi^2(K-1)^2}\right\}^{1/p}$$

$$\geqslant \left\{\frac{1}{2}\frac{K-1}{(p+1)(1+\pi^2(K-1)^2)}\right\}^{1/p},$$

which proves the lemma.

LEMMA 3 *Let $\mathcal{F}$ be the family of all subharmonic functions $u(z)$ in the disk $C(R)$, $R>2$ such that $u(0)=0$ and*

$$\sup_{|z|\leqslant R_1} u(z) \leqslant A < \infty \tag{42}$$

for some $R_1 \in (2, R)$. Then for every $q>0$ we have

$$\sup_{u\in\mathcal{F}} \|u(e^{i\theta})\|_q \leqslant C(q, A). \tag{43}$$

Proof Denote by μ the Riesz measure associated with a function $u\in\mathcal{F}$, $n(t,\mu)=\mu(\{|z|\leqslant t\})$. Jensen's inequality and (42) imply

$$n(2,\mu) \leqslant A_1 < \infty. \tag{44}$$

Here and in the following we denote by A_i different constants depending only on A in (42). Using the Poisson–Jensen formula in the disk $C(2)$ we obtain

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(2e^{i\varphi})\frac{3}{5-4\cos(\varphi-\theta)}\,d\varphi + \int_{C(2)} \log\left|\frac{2(z-\zeta)}{4-z\bar{\zeta}}\right| d\mu_\zeta$$

$$= \frac{3}{2\pi}\int_0^{2\pi}\frac{u(2e^{i\varphi})\,d\varphi}{5-4\cos(\varphi-\theta)} + \int_{C(2)}\log\frac{2}{|4-z\bar{\zeta}|}\,d\mu_\zeta + \int_{C(2)}\log|z-\zeta|\,d\mu_\zeta$$

$$= u_1+u_2+u_3, \qquad z=e^{i\theta}.$$

Using (42), (43) we get

$$|u_1(e^{i\theta})| \leqslant A, \qquad |u_2(e^{i\theta})| \leqslant n(2,\mu)\log 3 \leqslant A.$$

Applying the Hölder inequality we obtain ($q>1$):

$$\|u_3(e^{i\theta})\|_q^q \leqslant \frac{1}{2\pi}\int_0^{2\pi} d\theta\left\{\int_{C(2)} |\log|e^{i\theta}-\zeta||\,d\mu_\zeta\right\}^q$$

$$\leqslant n(2,\mu)^{q-1}\frac{1}{2\pi}\int_{C(2)} d\mu_\zeta\int_0^{2\pi}|\log|e^{i\theta}-\zeta||^q\,d\theta \leqslant A(q).$$

If $q\leqslant 1$ then (43) is evident.

Proof of Theorem 3 Consider the family of subharmonic functions

$$u_t(z)=\frac{\log|f(tz)|}{m_p(t,f)}, \qquad t\geq 1.$$

Without loss of generality suppose that $f(0)=1$, so that by Lemma 2 and (12) the family u_t satisfies the conditions of Lemma 3 with the constant A depending on p and K. Thus by (43)

$$\limsup_{r\to\infty}\frac{m_q(r,0,f)}{m_p(r,\infty,f)}=\limsup_{t\to\infty}\|u_t^-(e^{i\theta})\|_q\leq\sup_{t\geq 1}\|u_t^-(e^{i\theta})\|_q<C(p,q,K)<\infty$$

and the theorem is proved.

Remark Let $T:[1,\infty)\to\mathbb{R}$ be a continuous increasing function $C_1,C_2>1$. The value r is called (C_1,C_2)-normal if

$$T(C_1r)\leq C_2T(r).$$

W. Hayman [8] proved the following

LEMMA H *The lower logarithmic density of the set of (C_1,C_2)-normal values is at least $1-\rho(\log C_1)/\log C_2$ where ρ is the order of T.*

Our proof of Theorem 3 combined with Lemma H show that an arbitrary entire function f of finite order ρ satisfies

$$\frac{m_q(r,0,f)}{m_p(r,\infty,f)}\leq C(\rho,p,q,\varepsilon), \qquad \text{as} \quad r\to\infty$$

avoiding a set of upper logarithmic density ε. Here p and q are arbitrary positive numbers. On the other hand using Theorem 2 one can construct an entire function f of prescribed order ρ satisfying

$$m_p(r,\infty,f)=0(m_p(r,0,f)),$$

as $r\to\infty$ in the set of upper density 1.

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References

[1] J. M. Anderson, Asymptotic values of meromorphic function of smooth growth, *Glasgow Math. J.* **20** (1979), 155–162.
[2] T. Carleman, Extension d'un théorème de Liouville, *Acta Math.* **48** (1926), 363–366.
[3] D. Drasin and D. Shea, Pólya peaks and the oscillation of positive functions, *Proc. Amer. Math. Soc.* **34** (1972), 403–411.

[4] A. E. Eremenho and M. L. Sodin, О поведении целой функции на последовательности концентрических окружностей, в книге: Анализ в бесконечномерных пространствах и теория операторов. Сборник научных трудов, Киев, Наукова думка, 1983, 68-76.
[5] A. E. Eremenho and M. L. Sodin, О росте и убывании целых функций конечного порядка, Рукопись, депонированная в УкрНИИНТИ, № 4199и – Д83, 1-12.
[6] A. A. Goldberg and A. E. Eremenko, On asymptotic curves of entire functions of finite order, *Math. U.S.S.R. Sb.* **37** (1980), 509–533.
[7] W. K. Hayman, Slowly growing integral and subharmonic functions. *Comment. Math. Helvet.* **34** (1960), 75–84.
[8] W. K. Hayman, On the characteristic of functions meromorphic in the plane and their integrals, *Proc. London Math. Soc.* (*3*), **14a** (1965), 93–128.