In the definition of a (dis-)connected set E one can replace the requirement that $A \cap B = \emptyset$ by the weaker requirement that $A \cap B \cap E = \emptyset$.

(This is actually the definition in the book. But the two definitions are equivalent, as the following simple argument shows.)

Theorem. *Let E be a set in R^n. Suppose that A and B are open,*

$$A \cap E \neq \emptyset, \quad B \cap E \neq \emptyset, \quad E \subset A \cup B, \tag{1}$$

and

$$A \cap B \cap E = \emptyset. \tag{2}$$

Then one can find two other open sets A_1 and B_1 such that all properties (1) are satisfied by A_1 and B_1, and (2) is replaced by a stronger property

$$A \cap B = \emptyset. \tag{3}$$

Proof. Let $x \in A \cap E$. Then there exists $r(x) > 0$ such that

$$B(x, r(x)) \cap B \cap E = \emptyset. \tag{4}$$

Here $B(x, r)$ is the open ball of radius r. Indeed, if this is not so, then one can find a sequence $y_n \in B \cap E$ which tends to x. Then, because A is open, all y_n with sufficiently large n belong to A, and this contradicts (2). This proves the existence of $r(x)$ such that (4) holds.

Similarly, for every $y \in E \cap B$, there exists $r(y) > 0$ such that

$$B(y, r(y)) \cap A \cap E = \emptyset. \tag{5}$$

Now put

$$A_1 = \bigcup_{x \in A \cap E} B(x, r(x)/3), \quad \text{and} \quad B_1 = \bigcup_{y \in B \cap E} B(y, r(y)/3).$$

These sets are open, as unions of open balls, they both intersect E and their union contains E. It remains to show that their intersection is empty. Suppose that this is not the case. Then some ball $B(x, r(x)/3)$ intersects some ball $B(y, r(y)/3)$. Suppose, without loss of generality, that $r(x) \geq r(y)$. Then $\|x - y\| \leq r(x)/3 + r(y)/3 < r(x)$, so $B(x, r(x))$ contains y, but this contradicts the definition of $r(x)$, see (4), because $y \in B \cap E$. If $r(y) \geq r(x)$, one argues similarly using (5) instead of (4).

This ends the proof.

Remark. The argument works in any metric space, not only in R^n.