

Simultaneous stabilization, avoidance and Goldberg's constants

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Dedicated to the memory of A. Goldberg and V. Logvinenko

Abstract

This is an exposition for mathematicians of some unsolved problems arising in control theory of linear time-independent systems.

The earliest automatic control devices that I know are described in the book of Hero of Alexandria “Pneumatica”, see Fig. 1. In the modern times these devices are omnipresent (almost every home appliance contains at least one, a car has several, an airplane or a guided missile has many; an ingenious mechanical steering device of a sailboat permits you to sleep and to dine during your voyage, while it keeps prescribed direction with respect to the wind; one can add many other examples).

The mathematical theory of these devices begins, as far as I know, with George Biddell Airy (of the Airy function), Astronomer Royal, who investigated mathematically stabilization of the clockwork mechanism directing his equatorial.¹ The stability condition that “all poles must be in the left halfplane” was explicitly stated for the first time by J. C. Maxwell [20]. Parallel research was done in Eastern Europe by Aurel Stodola (1894) and Ivan

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¹Equatorial is a device that continuously adjusts a telescope direction to compensate for the diurnal rotation of the Earth. One of the most complicated modern control systems directs the Hubble telescope. It has to keep the direction of the telescope with high accuracy and for long time.

Vyshnegradsky (1877) who pioneered the use of complex function theory, anticipating the work of Nyquist (1932), see for example, [16].

Most of the XIX century research in the area was related to stabilizing the system which consists of a steam engine controlled by the governor.

These investigations led to the famous criteria in terms of coefficients of a polynomial for all its roots to belong to the left half-plane, (E. Routh, 1877, A. Hurwitz, 1895), see [10].

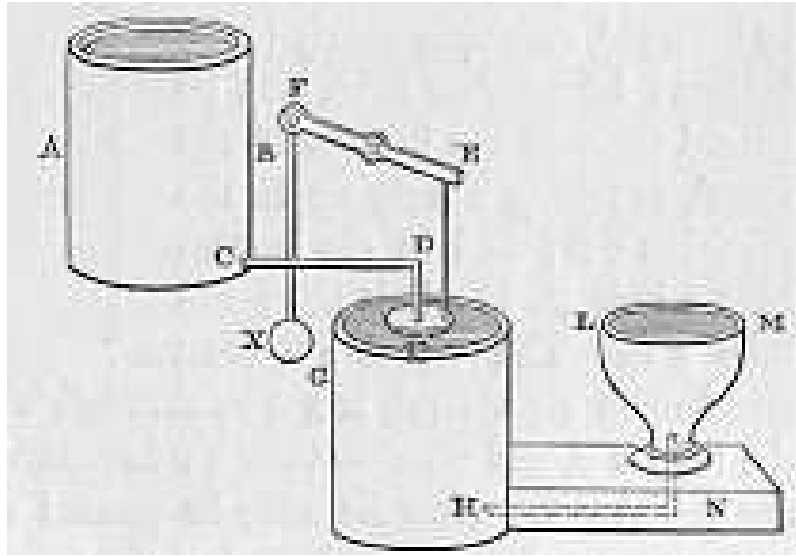


Fig. 1 A XIX century illustration made according to the description in the book of Hero Pneumatica.

A *linear system* of the simplest kind is described by 3 real matrices: (A, B, C) of sizes $n \times n$, $n \times m$ and $p \times n$ respectively. We have vectors depending on time: the *inner state* $x(t)$ with values in R^n , the *input* $u(t)$ with values in R^m and the *output* $y(t)$ with values in R^p . These are related in the following way:

$$\begin{aligned} x' &= Ax + Bu, \\ y &= Cx. \end{aligned}$$

We will only consider the case $m = p = 1$ (so called single input – single output systems).

Taking Laplace transforms, and assuming that $x(0) = u(0) = y(0) = 0$, we obtain $zX(z) = AX(z) + BU(z)$, $Y(z) = CX(z)$, so

$$Y(z) = C(zI - A)^{-1}BU(z) = p(z)U(z). \quad (1)$$

The rational function $p(z) = C(zI - A)^{-1}B$ is called the *transfer function*. It is real and $p(\infty) = 0$. Rational functions satisfying $p(\infty) = 0$ are called *proper*. For every proper rational p function there exists a triple (A, B, C) so that $p(z) = C(zI - A)^{-1}B$.

The correspondence between triples of matrices and rational functions is not trivial, not bijective, and there is a large literature on recovery of A, B, C from the transfer function (realization theory). But all essential properties of the system are encoded in the transfer function and here we identify a linear system with its transfer function.

Improper transfer functions are equally important, they arise from more general systems of differential equations with constant coefficients; I don't go into detail, but the primary object in this paper will be an *arbitrary* real rational function; we call it a transfer function. It completely describes a linear system.

A system is called *stable* if the transfer function has no poles in the open right half-plane H . The poles of the transfer function are nothing but the eigenvalues of the matrix A of the system.

For a given (maybe unstable) system, one may wish to *stabilize* it by attaching a feedback *controller*. A controller is a linear system of the same kind; it is described by another real rational transfer function $c(z)$. Attaching a controller as in the third diagram in Fig. 2 means that we take the output of our original system, transform it by the controller, and then add to the input:

$$Y = p(U + cY) = pU + pcY.$$

We obtain a new system, which is called the *closed loop system*. By solving with respect to Y we get the *closed loop transfer function*:

$$\frac{p}{1 - cp}. \quad (2)$$

Cancellation between poles and zeros of c and p is possible here, but engineers naturally do not want to rely on such cancellation. So they give the following definition:

A controller c *internally stabilizes* p if $1 - cp$ has no zeros in the right half-plane H , the poles of c are disjoint from the zeros of p in H , and the zeros of c are disjoint from the poles of p in H .

From now on by “stabilization” we mean “internal stabilization”. One can easily show that internal stabilization is equivalent to the condition that all four transfer functions

$$pc/(1 - pc), \quad c/(1 - pc), \quad p/(1 - pc), \quad 1/(1 - pc)$$

are without poles in H .

All these four transfer functions can be realized by attaching the feedback in various ways, as shown in Fig. 2 below.

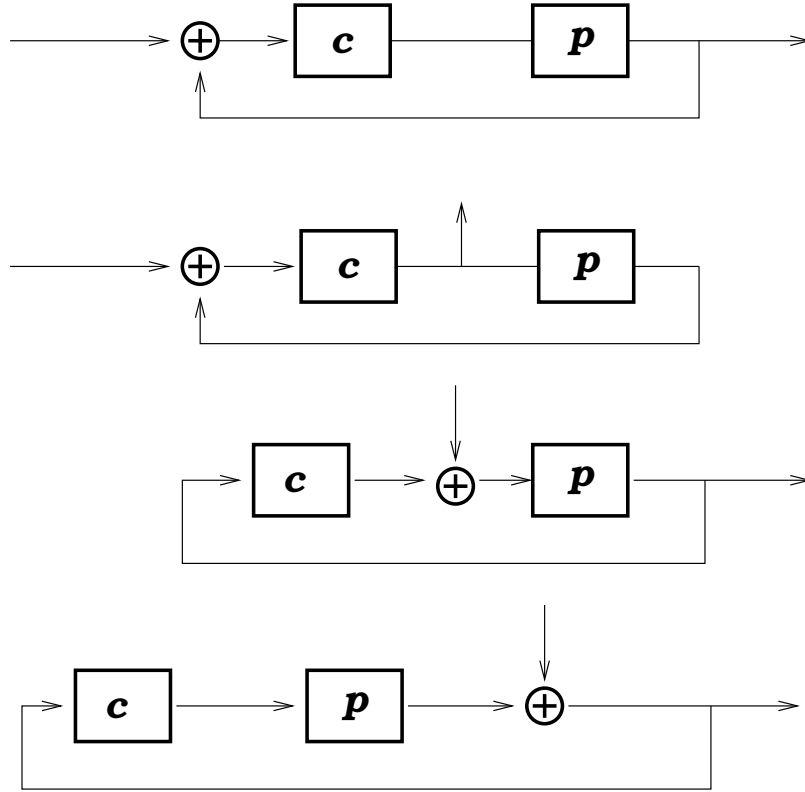


Fig. 2 Configurations corresponding to $pc/(1 - pc)$, $c/(1 - pc)$, $p/(1 - pc)$, $1/(1 - pc)$.

Here is another elegant way to rewrite the internal stabilization condition: c internally stabilizes p iff c avoids $1/p$ in the sense that

$$c(z) \neq 1/p(z), \quad z \in H. \quad (3)$$

Thus the *stabilization problem* is: for a given rational function p find a rational function c so that (3) holds.

We obtain an equivalent problem when the right half-plane H is replaced by the unit disc D . From the point of view of system theory, the unit disc setting corresponds to discrete-time systems. Instead of a differential equation, we have a recurrence relation,

$$x(n+1) = Ax(n) + Bu(n), \quad y(n) = Cx(n),$$

and in place of the Laplace transform we use the generating function $X(z) = \sum_{-\infty}^{\infty} x(n)z^n$. Then the transfer function $p(z)$ becomes $C(z^{-1}I - A)^{-1}B$, exactly as in the case of continuous time, and the system is stable if p has no poles in the unit disc, which now is equivalent to saying that A has no eigenvalues whose absolute value is greater than 1.

Stabilization of one system is always possible if one does not restrict the degree of c .

Now we consider simultaneous stabilization of several systems by one controller. The problem has evident practical meaning: the system that we want to stabilize may work in several different regimes (think of the cooling/heating system in your home, which is usually controlled by a single device, or an airplane during take off/landing/horizontal flight), mathematically this means that we want a single controller to stabilize several systems.

Consider the problem of stabilizing two systems p_1 and p_2 . This is equivalent to stabilization of one system by a *stable controller* [3].

Stability of the controller is a desirable property by itself: if the system p suddenly stops working, we don't want the controller to destroy itself.

A rational function avoiding two different rational functions p_i in H always exists, but it may have complex coefficients, even if p_i are real. One usually needs a controller with real coefficients. There is an obvious topological obstruction to the existence of real rational function without poles on $\mathbf{R}_{>0}$ which avoids a given real rational function. In fact, this is the only obstruction:

For a given real rational function p , there exists a real rational function c satisfying (3) and without poles in H if and only if p has even number of poles between every two adjacent zeros on $\mathbf{R}_{>0}$.

This neat statement is due to Youla, Bongiorno and Lu [28], and control theorists are very fond of it [3].

Now we consider simultaneous stabilization of three systems.

Finding a function which avoids three given functions is an interesting problem which attracted attention of pure mathematicians who were unaware of its application to control theory. It seems that the problem was stated for the first time in [25], and a connection with an “interpolation problem” of the kind stated below in Theorem 1 was established.

In [17] this avoidance problem is considered for meromorphic functions which avoid given rational functions in an arbitrary given region. The author credits Volberg and Eremenko who stated the problem and obtained some partial results. Apparently they were motivated by the “Lambda-lemma” and holomorphic motions which were discovered about that time [19, 18], [26], [9]. The lambda-lemma says that if finitely many meromorphic functions avoid each other in a disc, that is, if their graphs are disjoint, then one can always find an additional function which avoids all of them.

The main conclusion in [17] is that in any region one can always avoid two functions, but in general one cannot avoid three. This is easy to explain for the case of avoidance of three rational functions in \mathbf{C} . Take the avoided functions to be 0 , ∞ and z . If a meromorphic function f avoids them, then it must be rational, by Picard’s Theorem, but a rational function f that avoids 0 and ∞ in \mathbf{C} must be constant, so it cannot avoid z .

Similar results were obtained in control theory for the case of the unit disc or a halfplane.

We will work in the unit disc from now on. First we give a general reformulation of stabilization of three systems in terms of some unusual interpolation problem. Various special cases of this result are mentioned in the control literature, but I could not find a general statement.

Theorem 1. *Let ϕ_1, ϕ_2 and ϕ_3 be three rational functions without common poles, and suppose that the set*

$$E = \{z \in D : \phi(z) = \phi_2(z) = \phi_3(z)\} \quad (4)$$

is empty.

There exists a rational function f which avoids ϕ_i in U if and only if there exists a rational function g with the properties:

- (i) divisor of zeros of g coincides with the divisor of zeros of $\phi_3 - \phi_2$;*
- (ii) divisor of poles of g coincides with the divisor of zeros of $\phi_3 - \phi_1$, and*

(iii) *divisor of ones of g coincides with the divisor of zeros of $\phi_1 - \phi_2$.*

Condition that the ϕ_i have no common poles is added only for simplicity of formulation: the whole situation is invariant with respect to composition with fractional-linear transformations. Condition that $E = \emptyset$ holds for generic ϕ_i .

The correspondence between f and g is given by the cross-ratio

$$g = \frac{(f - \phi_1)(\phi_3 - \phi_2)}{(f - \phi_2)(\phi_3 - \phi_1)}.$$

In the case that there are triple intersections, that is $E \neq \emptyset$, one has to add the condition for each point $a \in E$:

$$g(z)(\phi_3(z) - \phi_1(z))/(\phi_3(z) - \phi_2(z)) = 1 + O(z^k), \quad z \rightarrow a, \quad (5)$$

where k is the order of the zero of $\phi_1 - \phi_2$ at a .

Thus, simultaneous stabilization of three functions (and the problem of avoidance of three functions) is equivalent to finding a function with prescribed zeros, ones and poles in the unit disc, counting multiplicity, and prescribed jets at finitely many points.

Interestingly, Nevanlinna [22] proposed a similar problem for meromorphic functions in \mathbf{C} : to find necessary and sufficient conditions that zeros, poles and 1-points of a meromorphic functions must satisfy. Some necessary conditions are known [22, 27], see also [25, 23]. Most of these results are for meromorphic functions in the plane.

Consider the following examples.

1. (Blondel [3]) For which δ the following three transfer functions in the unit disc are simultaneously stabilizable:

$$p_1 = z^2/(z - \delta), \quad p_2(z) = z^2/(z + \delta), \quad p_3(z) = 0 \quad ?$$

The stabilizer c has to be a rational function without poles in the unit disc avoiding $1/p_i$, $i = 1, 2$. This means $g(z) = z - c(z)z^2$ has to satisfy

$$g(0) = 0, \quad g'(0) = 1, \quad g(z) \neq \pm\delta, \quad |z| < 1.$$

According to a result of Bermant [2] this is possible if and only if

$$\delta \geq \delta_0 := 8\pi^2/\Gamma^4(1/4),$$

and the extremal function is not rational. This inequality gives a necessary condition of simultaneous stabilizability. Then an easy approximation argument shows that p_1, p_2 and p_3 are simultaneously stabilizable if and only if $\delta > 8\pi^2/\Gamma^4(1/4)$.

2. (Patel [24]) For which $a > 0$, the following three transfer functions in the unit disc are simultaneously stabilizable:

$$p_1(z) = z, \quad p_2(z) = z^2/(z - a), \quad p_3(z) = 0 \quad ?$$

The stabilizer c has to be a rational function without poles in the unit disc satisfying

$$c(z) \neq 1/z, \quad c(z) \neq (z - a)/z^2, \quad |z| < 1.$$

Introducing $g = (z - c(z)z^2)/a$ we rewrite this in the equivalent form:

$$g(z) \neq 1, \quad g(0) = 0 \leftrightarrow z = 0, \quad g'(0) = 1/a. \quad (6)$$

The answer follows from a theorem of Caratheodory [7], [21], [12]

If a holomorphic function g in the unit disc satisfies (6) then

$$|a| \geq 1/16.$$

There is a real holomorphic function for which equality holds. This extremal function is not rational.

A similar result, but with a smaller constant, was obtained for the first time by Hurwitz [15].

Now a simple approximation argument shows that the above three systems are simultaneously stabilizable if and only if $a > 1/16$. This answers a question stated in [24].

Suppose that we wish to stabilize three transfer functions, one of which avoids another. The problem is equivalent to finding a rational function without zeros and poles in the unit disc, which avoids one rational function p . Such c is called a *bistable controller*. I am not aware of any practical application of this “bistability property” by itself, but the desire to control three systems with a single controller is reasonable as explained above. The problem now is to find necessary and sufficient conditions on a rational function p for the existence of c satisfying (3) and having no zeros and no poles in D . Blondel [3] calls this “one of the major unsolved problems of control theory”.

A special case of Theorem 1 above, previously established by Blondel, says that the problem is equivalent to:

Finding a rational function w without poles in D , so that 1-points of w in D and zeros of w in D are prescribed (with multiplicities).

We refer to [4, 5] and the references in [3] for some necessary conditions that zeros and 1-points must satisfy.

Only one universal restriction (independent of degree) which zeros, poles and 1-points of a rational function must satisfy is known. It was found by Goldberg [11] and later independently by Blondel.

To state Goldberg's result, we introduce some notation. Let F_0 be the class of all holomorphic functions f in the rings

$$\rho(f) < |z| < 1,$$

with the properties that $f(z) \notin \{0, 1, \infty\}$, and the indices (winding numbers) of the curve

$$\gamma(f) = \{f(z) : |z| = (1 + \rho(f))/2\} \quad (7)$$

about 0 and 1 are non-zero and distinct. Let F_4, F_3, F_2, F_1 be the subsets of F_0 which consist of polynomials, rational, holomorphic, and meromorphic functions in D , respectively, having finite pairwise distinct numbers of zeros, poles and 1-points. We have $F_4 \subset F_2 \subset F_1 \subset F_0$ and $F_2 \subset F_3 \subset F_1 \subset F_0$. The constants $\rho(f)$ are defined for $f \in F_j$, $1 \leq j \leq 4$ as

$$\rho(f) = \max\{|z| : f(z) \in \{0, 1, \infty\}\}.$$

Now we put

$$A_j = \inf\{\rho(f) : f \in F_j\}, \quad 0 \leq j \leq 4.$$

Evidently $A_0 \leq A_1 \leq A_3 \leq A_4$ and $A_0 \leq A_1 \leq A_2 \leq A_4$. Goldberg's theorem says that

$$0 < A_0 = A_1 = A_3 < A_2 = A_4.$$

Moreover, extremal functions exist for A_0 and A_2 but do not exist for A_1, A_3 and A_4 .

This result shows that if a holomorphic function in the unit disc has finite, non-zero, distinct numbers of zeros and 1-points, then these zeros and one points cannot lie very close together.

So we have two absolute constants $0 < A_0 < A_2$ which are called Goldberg's constants. The exact value of A_0 is known:

$$A_0 = \exp(-\pi^2/(\log(3 + 2\sqrt{2}))) \approx 0.003701599,$$

and for A_2 there are estimates

$$0.00587465 < A_2 \leq \mu \approx 0.0252896.$$

The constant μ and a function which corresponds to it are conjectured to be extremal for A_2 ; this function h is described in detail in [6], and we will give a short description below.

If the indices of the curve γ about 0 and 1 are prescribed to be N_0, N_1 , we obtain constants $A_0(N_0, N_1)$. One can obtain an exact value of $A_0(N_0, N_1)$, for any given $N_1 > N_0 > 0$, see [6].

Being unable to prove that $A_2 = \mu$, the authors of [6] showed that μ is the solution of a restricted extremal problem:

Theorem. *A necessary and sufficient condition for the existence of a holomorphic function f in the unit disc, having no poles, a single simple zero at a and a single multiple 1-point at $-a$ is that $|a| \geq \mu$. If $a = \mu$ this function is unique and transcendental. If $|a| > \mu$ there exists a polynomial f with the stated properties.*

Thus in the simplest case of one simple zero, one multiple 1-point and no poles, we have a necessary and sufficient condition for the existence of a rational function with prescribed zeros, 1-points and poles in the unit disc.

This can be restated as a necessary and sufficient condition for a stabilization problem as follows:

The transfer function

$$\frac{(z + a)^2}{z - a}$$

can be stabilized by a bistable controller if and only if $|a| > \mu$.

In more complicated cases, there is no hope for such simple conditions. For example, Blondel [3] states the following problem²:

For which $\delta > 0$ there exists a rational function which in the unit disc has no poles, a single simple zero at 0, and exactly two simple 1-points $\pm i\delta$?

It is known that there exists $\delta_0 > 0$, with the property that such function exists for $\delta \geq \delta_0$ and does not exist for $\delta < \delta_0$.

²He even offered a prize of 1 kg of fine Belgian chocolate for this problem. Nevertheless it is still wide open.

Evidently $\delta_0 \geq A_2$ and it is not difficult to show that this inequality is strict. The current world record [8] for the estimate from above seems to be $\delta_0 < 0.1148$. The best known lower estimate is 0.01450779. It can be obtained from the estimate in [14] of the minimal length of a closed hyperbolic geodesic in a twice punctured disc [6].

In conclusion, we sketch the definition of the function which is conjectured to be extremal for A_2 . The fundamental group Γ of $\mathbf{C} \setminus \{0, 1\}$ is a free group generated by simple loops A and B around 0 and 1. Let Γ' be the subgroup generated by A and B^2 . It is also a free group on two generators. Let $g : X \rightarrow \mathbf{C} \setminus \{0, 1\}$ be the covering map corresponding to this subgroup Γ' , so that Γ' is the fundamental group of X . One can show that X is a Riemann surface which is conformally equivalent to the twice punctured disc, and we can identify it with $D \setminus \{-\mu, \mu\}$ for some $\mu \in D$. Then g becomes a holomorphic function in D which has one simple zero, say at $-\mu$ and one double 1-point at μ . We conjecture that $A_2 = \mu$. One can express our function g in terms of solutions of a Lamé equation and modular functions.

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