Critical values of generating functions of totally positive sequences

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Parametrization of various classes of entire and meromorphic functions by critical values is interesting from the point of view of geometric theory of meromorphic functions [5, 10, 16] and is also useful in several questions of analysis [5, 6, 7, 11, 13, 15]. In this paper we find such a parametrization for a class of meromorphic functions which occurs in the theory of totally positive sequences.

We denote by $ASWE$ the class of meromorphic functions of the form

$$f(z) = e^{\sigma z} \frac{\prod_k (1 + z/a_k)}{\prod_k (1 - z/b_k)}, \tag{1}$$

where $\sigma \geq 0$, (a_k) and (b_k) are two increasing sequences of positive numbers, finite or infinite (possibly empty), and

$$\sum_k \left(\frac{1}{a_k} + \frac{1}{b_k} \right) < \infty. \tag{2}$$

This class coincides with the set of generating functions of one-sided totally positive sequences, a. k. a. Pólya frequency sequences, [1, 4], see also [9, Ch. 8]. A function of the form (1) has exponential type σ.

Let us denote by (x_k) the sequence of real critical points of f, where

$$k \in (\mathbf{Z} \cap (-p, q)) \backslash \{0\}, \quad \text{where} \quad -\infty \leq -p < 0 < q \leq +\infty, \tag{3}$$

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so that f has $p-1$ negative and $q-1$ positive critical points. We assume that this sequence (x_k) is increasing, each critical point is repeated according to its multiplicity, and that $x_{-1} < 0 < x_1$. We do not exclude the case $p = q = 1$ when the sequence (x_k) is empty. Let $c_k = f(x_k)$ be the corresponding sequence of critical values.

A function $f \in ASWE$ will be called *normalized* if $f'(0) = 1$. We denote by $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ the one-point compactification of the real line. Our main result is the following.

Theorem 1 *All critical points of a function $f \in ASWE$ are real. For a sequence (c_k), $c_k \in \overline{\mathbf{R}}$, $-p < k < q$, $k \neq 0$, to be the sequence of critical values of a function $f \in ASWE$, it is necessary and sufficient that the following two conditions be satisfied:*

$$(-1)^k c_k \geq 0, \quad \textit{for} \quad -p < k < q, \quad k \neq 0, \quad c_k \neq \infty, \tag{4}$$

$$\textit{if} \quad c_k = 0 \quad \textit{then} \quad k < 0 \quad \textit{and} \quad 0 \in \{c_{k-1}, c_{k+1}\}, \tag{5}$$

$$\textit{if} \quad c_k = \infty \quad \textit{then} \quad k > 0 \quad \textit{and} \quad \infty \in \{c_{k-1}, c_{k+1}\}, \tag{6}$$

and

$$|c_{-k}| < |c_k|, \quad \textit{for} \quad 0 < k < r = \min\{p, q\}. \tag{7}$$

The correspondence between sequences (c_k) satisfying (4) *and* (7) *and normalized transcendental functions $f \in ASWE$ is bijective.*

A similar bijective correspondence holds between finite critical sequences and rational functions in $ASWE$. It will be established in the process of proof of Theorem 1.

Proof of Theorem 1. The following proposition permits to use approximation by rational functions in the proof of Theorem 1.

Proposition 1 *Let F be a subset of $ASWE$, such that the set $\{f'(0) : f \in F\}$ is bounded. Then F is a normal family in the whole complex plane $\mathbf{C}$.*

Proof.

$$f'(0) = \sigma + \sum_k \left(\frac{1}{a_k} + \frac{1}{b_k} \right),$$

where all summands are non-negative. So from every sequence $f_n \in F$ we can select a subsequence, such that the zeros, poles and σ's for this subsequence will converge, and the limit sequences will satisfy (2). □

Every function $f \in ASWE$ can be approximated by rational functions $f_n \in ASWE$, uniformly with respect to the spherical metric, on every compact subset of $\mathbf{C}$. (For the exponential factor we can use $\exp(\sigma z) = \lim(1+\sigma z/n)^n$.)

Proposition 2 *All critical points of a function $f \in ASWE$ are real, and we have*

$$\ldots \leq x_{-2} \leq a_2 \leq x_{-1} \leq a_1 < 0 < b_1 \leq x_1 \leq b_2 \leq \ldots . \tag{8}$$

Proof. Suppose first that f is rational of degree d, and $f(\infty) \notin \{0, \infty\}$. Then f has d zeros on the negative ray and d poles on the positive ray. By Rolle's theorem f has at least $d-1$ positive and $d-1$ negative critical points, counting multiplicity. Thus all critical points are real, and (8) holds. The general case follows by approximation. □

As a corollary from (8) we obtain that critical values of every function f in $ASWE$ satisfy (4), (5) and (6).

We recall that a point $a \in \overline{\mathbf{C}}$ is called an *asymptotic value* of a transcendental meromorphic function f if there exists a path $\gamma : [0, 1) \to \mathbf{C}$, which is called an *asymptotic path to* a, such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to a$ as $t \to 1$.

Let a be an asymptotic value, and $B(a, \epsilon)$ the disc (with respect to the spherical metric) of radius ϵ centered at a. For every $\epsilon > 0$ we can choose a component D_ϵ of the set $f^{-1}(B(a, \epsilon))$ such that $D_{\epsilon_1} \subset D_{\epsilon_2}$ for $\epsilon_1 < \epsilon_2$, and $\cap_{\epsilon>0} D_\epsilon = \emptyset$. Any such choice $\epsilon \mapsto D_\epsilon$ defines a *transcendental singularity* of f^{-1} over a. The sets D_ϵ are called ϵ-neighborhoods of this singularity. A transcendental singularity is called *direct* if for some $\epsilon > 0$ we have $f(z) \neq a$ for $z \in D_\epsilon$. Otherwise it is called *indirect*.

We will use

Theorem A *For a meromorpic function f of order ρ, the inverse f^{-1} has at most* $\max\{1, 2\rho\}$ *direct singularities.*

This is due to Ahlfors, see [2] or [12].

Proposition 3 *For a transcendental function $f \in ASWE$, the only possible asymptotic values are* 0 *and* ∞*.*

Proof. Suppose that $a \notin \{0, \infty\}$ is an asymptotic value. We claim that there exists an asymptotic path to a or $\overline{a}$ in the (open) upper half-plane. If $a \notin \overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ this is evident, because $f(\mathbf{R}) \subset \overline{\mathbf{R}}$. If a is real, let γ be any asymptotic path to a. Then $f(z) \to a$ as $z \to \infty$, $z \in \gamma \cup \overline{\gamma}$. The last set is connected and symmetric with respect to the real axis, so we can find an asymptotic path in the upper half-plane. This proves the claim.

Let us choose now a simple asymptotic path γ to $a \notin \{0, \infty\}$ in the upper half-plane. Consider a simple curve $\Gamma = \gamma \cup \overline{\gamma} \cup \gamma_0$, where γ_0 is a bounded curve. Let G_+ and G_- be the components of $\mathbf{C} \backslash \Gamma$, and assume that γ_0 is chosen in such a way that G_+ contains all poles and G_- contains all zeros of f.

Notice that f cannot tend to a as $z \to \infty$ along the negative ray. Indeed f either has infinitely many zeros on the negative ray, or tends to 0 as $z \to \infty$ along the negative ray, which follows from (1). Applying Lindelöf's theorem [12] to G_- we conclude that f is unbounded in G_-. It easily follows that there exists an asymptotic path in G_- on which $f(z) \to \infty$.

By a similar argument, there exists an asymptotic path to 0 in G_+. The singularities of f^{-1} corresponding to these two paths are direct because $f(z) \neq \infty$ in G_- and $f(z) \neq 0$ in G_+.

Let γ_1 be a curve in the intersection of G_+ with the upper half-plane, such that $f(z) \to 0$, $z \in \gamma_1$. Let γ_2 be a bounded curve such that $\Gamma_2 = \gamma \cup \gamma_1 \cup \gamma_2$ is a simple curve in the upper half-plane. Let G be the component of $\mathbf{C} \backslash \Gamma_2$ which is contained in the upper half-plane. By appropriate choice of γ_2 we can also achieve that $G \subset G_+$. Applying Lindelöf's theorem to G we conclude that f is unbounded in G, so there exists an asymptotic path in G on which $f(z) \to \infty$. The singularity of f^{-1} corresponding to this path is also direct because f has no poles in G.

Thus we found three direct singularities, which contradicts Theorem A because f is of order 1. This proves Proposition 3. $\square$

Following Vinberg [16] we introduce the *net* $\Gamma = f^{-1}(\overline{\mathbf{R}})$. By Propositions 2 and 3, all critical and asymptotic values of f belong to $\overline{\mathbf{R}}$, so the restriction $f : \mathbf{C} \backslash \Gamma \to \overline{\mathbf{C}} \backslash \overline{\mathbf{R}}$ is a covering map. Thus each component D of $\mathbf{C} \backslash \Gamma$ is a simply connected region in $\mathbf{C}$, which is mapped homeomorphically onto one of the half-planes $\mathbf{C} \backslash \mathbf{R}$. If we denote by ∂D the space of prime ends of D, then the induced map $f : \partial D \to \overline{\mathbf{R}}$ is a homeomorphism for each D, because every conformal homeomorphism between open discs extends to a homeomorphism of their closures. We will call these components D the *faces* of the net. The set $\Gamma \backslash \{\text{critical points}\}$ is a disjoint union of simple analytic curves. It is clear that these curves have well-defined ends which can be critical points or ∞. These curves will be called the *edges* of the net.

It is clear that $\mathbf{R} \subset \Gamma$ and that Γ is symmetric with respect to $\mathbf{R}$. We are going to describe all possible nets up to the following equivalence relation: $\Gamma_1 \sim \Gamma_2$ if $\Gamma_1 = \phi(\Gamma_2)$, where ϕ is a homeomorphism $\phi : \mathbf{C} \to \mathbf{C}$ such that $\phi(0) = 0$, $\phi(\overline{z}) = \overline{\phi(z)}$, and ϕ is increasing on the real line.

Let x be a critical point of multiplicity m. This means that x is repeated m times among the x_k as in (8), and that f is locally $(m+1)$-to-1 in a neighborhood of x. So exactly m edges γ in the upper half-plane meet at x. (The total number of edges meeting at x is $2m+2$.)

We divide all edges of the net in the upper half-plane into three types. An edge which connects two finite critical points is of the *first type*. An edge which connects a finite critical point to infinity is of the *second type*, and an edge whose both ends are at infinity is of the *third type*.

The following proposition describes all possible nets of a function f in $ASWE$.

Proposition 4 *Let Γ be a net of a function $f \in ASWE$, having $p-1$ negative and $q-1$ positive critical points counting multiplicity. Put $r = \min\{p, q\}$.*

a) If $p = q = \infty$ the net has only edges of the first type in the upper half-plane. These edges connect x_{-k} with x_k for each positive integer k.

b) If $r < \infty$ but $\max\{p, q\} = \infty$, the intersection of the net with the upper half-plane consists of $r-1$ edges of the first type, each of them connecting x_{-k} ray with x_k, and infinitely many edges of the second type.

c) If $p < \infty$ and $q < \infty$, the intersection of the net with the upper half-plane consists of:

$r-1$ edges of the first type, each of them connecting x_{-k} with x_k, for $1 \leq k \leq r-1$;

$\max\{p, q\} - r$ edges of the second type,

and in addition it may contain countably many edges of the third type. These edges of the third type are absent if and only if f is rational.

Corollary *An infinite sequence (c_k) determines the net completely. A finite sequence (c_k) determines two nets: one for a rational function and one for a transcendental one.*

Proof of Proposition 4. Let γ be an edge in the upper half-plane. Suppose that none of the endpoints of $\gamma \subset \Gamma$ is ∞, so the endpoints are critical points in $\mathbf{R}$.

We claim that one of these endpoints belongs to the positive ray and another to the negative ray. Suppose that this is not so, for example, let the endpoints $x_k \leq x_m$ be both positive. Consider the region G bounded by the Jordan curve $\gamma \cup [x_k, x_m]$. The closure of this region contains no zeros of f because all zeros are on the negative ray. But there is at least one face D such that $D \subset G$, and we conclude that $f(z) \neq 0$ for $z \in \partial D$, contradicting the fact that f maps ∂D onto $\overline{\mathbf{R}}$ homeomorphically. This proves our claim.

Now we show that edges of the second kind originating on the positive and negative ray cannot be simultaneously present. Assume the contrary. Let k be the smallest positive integer such that an edge of the second kind γ_k originates at x_k, and m the smallest positive integer such that an edge of the second kind γ_{-m} originates at x_{-m}. Then it is easy to see that $k = m$ and each critical point x_{-n} for $n < k$ is connected with the critical point x_n by an edge of the first kind. So both γ_k and γ_{-k} belong to the boundary of some face D. We have a pole b_k and a zero $-a_k$ in $\partial D \cap \mathbf{R}$, and in addition, $f(z)$ tends to 0 or ∞ as $z \to \infty$, $z \in \gamma_k$. This contradicts the fact that $f : \partial D \to \overline{\mathbf{R}}$ is a homeomorphism.

Now we can complete the proof in the case a), the case of doubly infinite sequence x_k. It is clear that there cannot be edges of the second kind in this case. Thus all negative critical points are connected to positive critical points by sedges of the first kind, so all faces are compact and there are no edges of the third kind.

Now consider the case b). We have finitely many (namely $r-1$) edges of the first kind and infinitely many edges of the second kind. To complete the picture, it remains to show that there are no edges of the third kind in this case. Suppose the contrary. Let G be a component of

$$\mathbf{C} \backslash \cup \{\text{edges of the first and second kind}\}$$

in the upper half-plane, such that G has an edge γ of the third kind on its boundary. It is clear that $\partial G \cap \mathbf{R}$ contains either a zero or a pole. Then γ is mapped onto its image in $\overline{\mathbf{R}}$ homeomorphically, so there are two asymptotic values, say a and b to which $f(z)$ tends as $z \in \gamma$, $z \to \infty$. If $a = b$ then $f(\gamma) = \overline{\mathbf{R}} \backslash \{a\}$, but this is impossible because f maps ∂G onto $\overline{\mathbf{R}}$ homeomorphically. If $a \neq b$ then $\{a, b\} = \{0, \infty\}$ by Proposition 3, and we obtain a contradiction again, because ∂G already contains a zero or a pole on the real line. This completes consideration of the case b).

Case c) can be studied similarly and this is left to the reader. □

Proposition 5 *For every $f \in ASWE$, the inequalities* (7) *hold.*

Proof. Suppose first that k is odd, $1 \leq k \leq r-1$. If x_{-k} is a multiple zero then $c_{-k} = 0$; if x_k is a multiple pole then $c_k = \infty$. In both cases there is nothing to prove. Now assume that $c_{-k} \neq 0$ and $c_k \neq \infty$. Our assumption that k is odd, together with (4), implies that

$$c_k < 0 \quad \text{and} \quad c_{-k} < 0.$$

So $x_{-k} \in (a_{k+1}, a_k)$, and $x_k \in (b_k, b_{k+1})$. Consider the arc

$$\Gamma = [a_{k+1}, x_{-k}] \cup \gamma \cup [x_k, b_{k+1}],$$

where γ is the edge of the net connecting x_{-k} with x_k in the upper half-plane.

This arc Γ is a part of the boundary of some face, so it is mapped homeomorphically onto some arc of $\overline{\mathbf{R}}$, namely on the negative ray, because $f(a_{k+1}) = 0$, $f(b_{k+1}) = \infty$, and $f(x_k) < 0$. It follows immediately that $f(x_k) < f(x_{-k}) < 0$ that is $|c_k| > |c_{-k}|$, as advertised.

The case of even k is completely similar. This proves (7). □

To complete the proof of Theorem 1, it remains to prove existence and uniqueness of a function $f \in ASWE$ with prescribed sequence of critical values satisfying (4) and (7).

Suppose first that p and q in (3) are finite. First we are going to construct a rational function $f \in ASWE$ with the sequence of critical values (c_k), where $-p < k < q$, $k \neq 0$. Similar constructions were used in [16], [6] and elsewhere.

It is easy to see that this sequence (c_k) defines the class of the sequence (x_k) modulo increasing homeomorphisms of the real line, fixing zero. Indeed, the maximal segments of equal x_k's correspond to the maximal segments of zeros or infinities in the sequence (c_k). Fix some sequence (y_k) in the class of (x_k). Consider the net Γ with vertices at y_k and no edges of the third kind. We assume that all edges are of finite length with respect to the spherical metric. The net is uniquely defined by the class of the sequence (x_k).

Using the net Γ and the sequence (c_k) we construct a open and discreet ("topologically holomorphic") map g from $\overline{\mathbf{C}}$ to itself ramified at y_k and possibly at infinity.

First we define g at the vertices of the net by putting $g(y_k) = c_k$. We also put $g(\infty) = \infty$ if there is an edge from the negative ray to ∞ and $g(\infty) = 0$ if there is an edge from the positive ray to ∞.

The following observation is crucial. Suppose that D is a face with four vertices on ∂D. Then the map g we just defined, which sends vertices to some points on the circle $\overline{\mathbf{R}}$ *respects the cyclic order*, that is either preserves or reverses it, depending on the choice of orientations of ∂D and $\overline{\mathbf{R}}$.

Indeed, suppose that the four vertices on ∂D are $x_{-k-1}, x_{-k}, x_k, x_{k+1}$, where the order corresponds to the natural orientation of ∂D. Then our map sends these points to $c_{-k-1}, c_{-k}, c_k, c_{k+1}$ which are in a cyclic order on $\overline{\mathbf{R}}$ because of the inequalities (4) and (7). Furthermore, $0 \in [c_{-k-1}, c_{-k}]$ and $\infty \in [c_k, c_{k+1}]$.

Then we extend g to the edges so that it maps the closure of an edge homeomorphically onto an arc of the circle $\overline{\mathbf{R}}$. There are two ways to choose this arc, and we use the following rules:

1. On the real line, $g(x) \neq \infty$ for $x < 0$, and $g(x) \neq 0$ for $x > 0$.
2. On all edges that are disjoint from the real line, $g(z) \notin \{0, \infty\}$.

These rules define g on Γ. It is continuous. It is easy to verify that for every face D, the map $g : \partial D \to \overline{\mathbf{R}}$ is a homeomorphism. This follows from our previous remark that restrictions of g on the boundary vertices of D respects cyclic order.

Suppose that ∂D is equipped with the standard orientation (so that D stays on the left), and $\overline{\mathbf{R}}$ with the increasing orientation. Then all faces are divided into two categories:
(1) those for which $g : \partial D \to \overline{\mathbf{R}}$ preserves the orientation and
(2) those for which it reverses the orientation.
If two faces have a common edge they belong to different categories.

We extend g homeomorphically into the interiors of faces, so that the faces of the type (1) are mapped into the upper half-plane, and faces of the second type into the lower half-plane.

The resulting map is topologically homeomorphic, and it is clear that it can be chosen with the following symmetry property: $g(\overline{z}) = \overline{g(z)}$.

Now there exists a homeomorphism ϕ, which is also symmetric and $f = g \circ \phi$ is a rational function, $f(0) = 1$.

Thus the existence statement in Theorem 1 is proved for the case of finite critical sequences and rational functions. To obtain the general case we approximate a given sequence by finite ones and refer to Proposition 1.

Now we prove the uniqueness statement. Suppose that two transcendental functions f_1 and f_2 of the class $ASWE$ have the same sequence (c_k). Then their nets are equivalent, and this implies that the "Riemann surfaces spread over the sphere" (see, for example [5, 3, 6, 16]) of f_1^{-1} and f_2^{-1} are isometric. (A Riemann surface spread over the sphere is the plane equipped with the pullback of the spherical metric via f). It follows that a branch of the map $f_1^{-1} \circ f_2$ maps conformally the plane with the critical points of f_2 deleted into the plane with the critical points of f_1 deleted. By the removable singularity theorem this maps has to be of the form $az + b$. Normalization conditions show that $a = 1$ and $b = 0$. This completes the proof of Theorem 1. □

Thus our class $ASWE$ has three different parametrizations: an analytic one as in (1), a geometric one by the sequence (c_k), and the third one, in terms of its Taylor coefficients, which are exactly the one-sided Pólya frequency sequences.

A necessary and sufficient condition for $\sigma = 0$ can be given in terms of the Taylor coefficients of f, see [9, Ch. 8, Thm. 10.1].

It is interesting to find a criterion for $\sigma = 0$ in (1) in terms of (c_k). In the symmetric case that $c_k = 1/c_{-k}$ an equivalent problem was posed by Teichmüller [14] and solved by A.A. Goldberg [8]. It is easy to extend Goldberg's result to the general case.

From now on we assume for simplicity that a function $f \in ASWE$ has infinitely many critical points.

This is equivalent to the property that the union of the sets of zeros and poles is infinite.

Definition *We say that a sequence $(c_k)_{k=1}^{\infty}$ satisfies condition K if*

$$\sum_{n=1}^{\infty} \frac{1}{\min_{k \geq n} \log^+ |c_k| + 1} < \infty.$$

Theorem 2. *Let $f \in ASWE$ be a transcendental meromorphic function given by equation (1). Then $\sigma > 0$ if an only if each of the two sequences $(c_k)_{k=1}^{\infty}$ and $(1/c_{-k})_{k=1}^{\infty}$ is either finite or satisfies condition K.*

Proof. Let f be defined by (1). Put

$$g(z)=\frac{f(z)}{f(-z)}=\exp(2\sigma z)\frac{\prod_{k=1}^{\infty}(1+z/r_k)}{\prod_{k=1}^{\infty}(1-z/r_k)}, \tag{9}$$

where (r_k) is the "union" of the sequences (a_k) and (b_k), $r_k>0$. This means that (r_k) is an increasing sequence, and the number of times some term occurs in (r_k) equals the sum of the numbers of times this term occurs in (a_k) and in (b_k). Evidently,

$$a_k \geq r_k \quad \text{and} \quad b_k \geq r_k, \tag{10}$$

whenever a_k or b_k is defined.

Thus $\sigma=0$ if and only if g is a Blaschke product. Let

$$A_k=\inf_{x\leq -a_k}\log\frac{1}{|f(x)|}=\inf_{m\geq k}\log\frac{1}{|c_{-m}|},$$

$$B_k=\inf_{x\geq b_k}\log|f(x)|=\inf_{m\geq k}\log|c_m|,$$

and

$$R_k=\inf_{x\geq r_k}\log|g(x)|.$$

We set $A_k=+\infty$ or $B_k=+\infty$ if a_k or b_k is not defined.

Notice that $R_k>0$. Goldberg's theorem [8, Thm. 10] says that g is a Blaschke product if and only if

$$\sum_{n=1}^{\infty}\frac{1}{R_n}=\infty. \tag{11}$$

Suppose that $\sigma=0$, so g is a Blaschke product. We want to prove that at least one of the two series

$$\sum_{k=1}^{\infty}\frac{1}{A_k^+} \quad \text{or} \quad \sum_{k=1}^{\infty}\frac{1}{B_k^+} \quad \text{diverges.} \tag{12}$$

If A_k or B_k does not tend to $+\infty$, this is evident. So assume that

$$A_k\to+\infty \quad \text{and} \quad B_k\to+\infty. \tag{13}$$

In view of (13) and (10) we have for k large enough:

$$\begin{aligned} R_k &= \inf_{x \geq r_k} \left(\log|f(x)| + \log \frac{1}{|f(-x)|} \right) \\ &\geq \max\left\{ \inf_{x \geq r_k} \log|f(x)|, \inf_{x \leq -r_k} \log \frac{1}{|f(x)|} \right\}. \end{aligned}$$

Now, (r_k) is the union of the sequences (a_k) and (b_k). If $r_k = a_{k'}$ for some k' then the previous estimate gives $R_k \geq A_{k'}$. If $r_k = b_{k'}$ then $R_k \geq B_{k'}$. Taking into account that $R_k > 0$ (because $|g(x)| > 1$ for $x > 0$ which follows from (9)), we obtain

$$\sum_{k=1}^{\infty} \frac{1}{R_k} \leq \sum_{k=1}^{\infty} \frac{1}{A_k^+} + \sum_{k=1}^{\infty} \frac{1}{B_k^+}.$$

So if (11) implies (12).

Now suppose that $\sigma > 0$. Using the fact that the canonical products in (1) are of minimal type, we obtain for $k > 0$

$$\log|c_k| = \log|f(x_k)| \geq (\sigma + o(1))x_k \geq (\sigma + o(1))b_k.$$

Now (2) implies that $(c_k)_{k=1}^{\infty}$ satisfies condition K. The proof is similar for $(1/c_{-k})_{k=1}^{\infty}$. □

Entire $ASWE$ functions constitute an important class which is called $LP1$, the first Laguerre-Pólya class, the closure of the set of real polynomials with negative zeros. Our results generalize the known results about $LP1$. Parametrization of $LP1$ by critical values (a special case of Theorem 1) is due to MacLane [10]. Another, more transparent proof was given by Vinberg [16] who introduced the nets. Theorem 2 is new even for the class $LP1$.

Another extension of the class $LP1$ is the class of entire functions $LP2$, the second Laguerre-Pólya class. It can be defined as the closure of the set of real polynomials with real zeros. According to a theorem of Pólya, functions $f \in LP2$ have a parametric representation

$$f(z) = z^m \exp(-\sigma z^2 + \tau z) \prod \left(1 - \frac{z}{a_k}\right) \exp \frac{z}{a_k}, \tag{14}$$

where a_k and τ are real, $a_k \neq 0$,

$$\sum_k \frac{1}{|a_k|^2} < \infty,$$

and $\sigma \geq 0$.

The theory of $LP2$ can be based on the following proposition analogous to Proposition 1.

Proposition 6 *Let $F \subset LP2$ be a subset with the property $f(0) = 1$ and $|f''(0) - (f')^2(0)| \leq M < \infty$ for $f \in F$. Then F is a normal family.*

To prove this one observes that for $f \in LP2$, the function g defined $g(z^2) = f(z)f(-z)$ belongs to $LP1 \subset ASWE$ so one can use Proposition 1.

□

Parametrization of $LP2$ by critical values was also obtained by MacLane in [10], and substantially simplified by Vinberg[1] in [16]. We recall this result.

Let us call two functions of the class $LP2$ equivalent if $f_1(z) = f_2(az+b)$, where $a, b \in \mathbf{R}$, $a \neq 0$.

All critical points of a function $f \in LP2$ are real. They form an increasing sequence (x_k) which can be infinite in both directions, or in one direction, or finite. Let $c_k = f(x_k)$ be the sequence of critical values. It is easy to see that the signs of (c_k) alternate.

It is easy to prove that the only possible asymptotic values of a function $f \in LP2$ are 0 and ∞. If the sequence of critical points is finite in one direction (this happens exactly when the sequence of zeros is finite in the same direction), the function tends to a limit, 0 or ∞ along the real line in this direction.

If this happens we extend the sequence (c_k) in this direction by an infinite sequence of 0's if the asymptotic value is 0 or $\pm\infty$'s if the asymptotic value is ∞. The signs of $\pm\infty$ are chosen in such a way that the resulting extended sequence alternates.

Let C be the set of all sequences of real numbers or symbols $\pm\infty$, with alternating sign and the following property: whenever a $\pm\infty$ occurs at some place it also occurs everywhere on the right or everywhere on the left of this place. Two sequences (c_k) and (c'_k) of the class C are called *equivalent* if $c_k = c'_{k+m}$ for some integer m.

MacLane–Vinberg theorem says that the correspondence

$$\text{function} \mapsto \text{extended sequence of critical values}$$

between equivalence classes of non-constant $LP2$ functions and equivalence classes of sequences in C is bijective.

[1]The statement of the result in [16] contains a minor mistake.

One can make this correspondence "explicit". For this purpose we associate to a sequence $c = (c_k) \in C$ the domain $D_c \subset \mathbf{C}$ which is obtained from the left half-plane by deleting the horizontal rays

$$\{t + \pi i k : -\infty < t < \log |c_k|\}.$$

If $c_k = \pm\infty$, this "ray" becomes a line, and our cuts break the plane into infinitely many regions. In this case we take for D_c that region which is not a horizontal strip of width π, if such region exists. If it does not exist, we take any of the strips.

Let $\theta_c : H \to D_c$ be a conformal map of the upper half-plane onto D_c with sends $\infty \in \partial H$ to the prime end of D_c at infinity, corresponding a the ray $[x_0, \infty) \subset \mathbf{R}$, where $x_0 \in D_c$. This map is defined up to a composition with an automorphism $z \mapsto az + b$ of H, where $a \in \mathbf{R}^*$ and $b \in \mathbf{R}$.

By the Symmetry Principle,

$$f_c(z) = \exp \theta_c(z)$$

has an analytic continuation to the whole plane. Thus f is an entire function, and the MacLane–Vinberg correspondence is given by $c \mapsto f_c$.

The following analog of our Theorem 2 holds for $LP2$.

Theorem 3 *Let f be a function with infinitely many zeros defined by* (14). *Let $c = (c_k)_{k \in \mathbf{Z}}$ be the extended sequence of its critical values. Then $\sigma > 0$ in* (14) *if and only if both sequences $(c_k)_{k=1}^{\infty}$ and $(c_{-k})_{k=1}^{\infty}$ satisfy condition K.*

We only sketch the proof. First we define a function g by the equation $g(z^2) = f(z)f(-z)$. It is easy to see that $g \in LP1$. By an argument similar to that in the proof of Theorem 2, the critical points of g satisfy condition K if and only if both sequences $(1/c_k)_{k=1}^{\infty}$ and $(1/c_{-k})_{k=1}^{\infty}$ of critical points of f satisfy condition K.

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