Math 525, Midterm exam, Spring 2014

1. Let f be a non-constant analytic function. Can the function $|f|$ be harmonic? Justify your answer.

Solution 1. $|f| = \sqrt{u^2 + v^2}$. To see whether this can be harmonic, one has to compute the Laplacian. But of course you have to compute it correctly.

$$|f|_x = (u^2 + v^2)^{-1/2}(uu_x + vv_x),$$

$$|f|_{xx} = (u^2 + v^2)^{-3/2}\left(-(uu_x + vv_x)^2 + (u^2 + v^2)(u_x^2 + uu_{xx} + v_x^2 + vv_{xx})\right).$$

Similarly for the derivatives in y. When we add these, using $u_{xx} + v_{xx} = 0$, after simplification we obtain

$$|f|_{xx} + |f|_{yy} = (u^2 + v^2)^{-3/2}\left((uv_x - vu_x)^2 + (uv_y - vu_y)^2\right).$$

Therefore we must have

$$uv_x - vu_x = 0, \quad uv_y - vu_y = 0.$$

Now we can use that u, v satisfy the Cauchy–Riemann conditions. Considering the last two equations as equations with respect to u, v with coefficients v_x, u_x, v_y, u_y, we see using Cauchy-Riemann condition that the determinant is $u_x^2 + u_y^2$. This can be zero only at isolated points (because f is not constant). At other points, the determinant is not zero, so we conclude that $u = v = 0$ at those other points. This cannot happen, so $|f|$ cannot be harmonic.

Solution 2. Let $v(z) = |z| = \sqrt{x^2 + y^2}$. Then $u(z) = v(f(z))$. And at those points where $f'(z) \neq 0$ we have an inverse function so $u(f^{-1}(z)) = v(z)$. If u is harmonic then v must be harmonic. But v is not harmonic, except at $z = 0$:

$$v_x = (x^2 + y^2)^{-1/2}x,$$

$$v_{xx} = (x^2 + y^2)^{-3/2}y^2 > 0,$$

and similarly for the derivative with respect to y.

Solution 3 (the simplest one). If f has zeros, $|f|$ cannot be harmonic by the minimum principle. If f has no zeros, then $u = \log|f|$ is harmonic.

Then if $|f|$ is harmonic we will have a harmonic function u such that e^u is also harmonic. Computing the Laplacian, we obtain

$$(e^u)_x = e^u u_x,$$

$$(e^u)_{xx} = e^u(u_x^2 + u_{xx}).$$

Writing this for y and adding and using $u_{xx} + u_{yy} = 0$, we obtain $u_x^2 + u_y^2 = 0$, so u is constant, contradiction!

(Congratulations to one student who found this solution).

2. Find all solutions of the equation $\sin z = 3$ and sketch them in a picture.

Solution. $\sin z = (e^{iz} - e^{-iz})/(2i)$. setting $w = e^{iz}$ we obtain a quadratic equation

$$w - 1/w = 6i,$$
$$w^2 - 6iz - 1 = 0,$$

so

$$w_{1,2} = (3 \pm 2\sqrt{2})i.$$

Now take $\log u = \mathrm{Log}\,|u| + i \arg u$, and notice that $3 \pm 2\sqrt{2} > 0$, and $\mathrm{Arg}\, i = \pi/2$, so

$$iz = \mathrm{Log}\,(3 \pm 2\sqrt{2}) + \pi i/2 + 2\pi i k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Thus

$$z = -i\mathrm{Log}\,(3 \pm 2\sqrt{2}) + \pi/2 + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

To make a correct picture notice that $(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 1$, therefore $\mathrm{Log}\,(3 + 2\sqrt{2}) = -\mathrm{Log}\,(3 - 2\sqrt{2})$, so we have two horizontal rows of points with imaginary part $\pm\mathrm{Log}\,(3 + 2\sqrt{2})$ and real part in arithmetic progression $\pi/2 + 2\pi k$.

3. For every integer m (positive, negative, zero) compute the integral

$$\int_{|z|=2} \overline{z}^m dz,$$

where the circle is oriented counterclockwise.

Solution 1. The function $\overline{z}$ is NOT ANALYTIC! So we cannot apply to it any theorem proved for analytic functions. But the integral is easy to compute by definition. The parametrization of the circle is $z(t) = 2e^{it},\ 0 \leq t < 2\pi$. So

$$\overline{z}^m = 2^m e^{imt}, \quad dz = 2ie^{it}dt,$$

and

$$\int_{|z|=2} \overline{z}^m dz = \int_0^{2\pi} 2^{m+1} ie^{(1-m)it} dt.$$

This is zero unless $m = 1$. If $m = 1$, this is $2\pi \times 2^{m+1}i = 8\pi i$.

Solution 2. One can reduce to analytic functions by writing $\overline{z} = |z|^2/z$. Then

$$\int_{|z|=2} \overline{z}^m dz = 2^{2m} \int_{|z|=2} z^{-m} dz.$$

The last integral is 0 unless $-m = -1$. When $m = 1$, the last integral is $2\pi i$ and we obtain the answer $8\pi i$.

4. Evaluate the integral

$$\int_{|z-1|=1} \frac{\cos z}{z^2-1} dz,$$

where the circle is oriented counterclockwise.

Hint: apply Cauchy's integral formula to an appropriate function.

Solution. Following the hint, we choose

$$f(z) = \frac{\cos z}{z+1}.$$

Then our integral equals

$$\int_{|z-1|=1} \frac{f(z)dz}{z-1} = 2\pi i f(1) = \pi i \cos 1,$$

because f is analytic inside and on the contour of integration.

5. Find a bounded harmonic function in the first quadrant which takes the boundary values 1 on the positive imaginary ray, and -1 on the positive real ray.

Solution.

$$\frac{4}{\pi}\operatorname{Arg} z - 1.$$

6. Find all values of i^i.

Solution. By definition,

$$i^i = e^{i \log i} = e^{i(\operatorname{Log}|i| + i \operatorname{Arg} i + 2\pi i k)}.$$

As $\operatorname{Log}|i| = 0$ and $\operatorname{Arg} i = \pi/2$, we obtain

$$e^{-\pi/2+2\pi k}, \quad k = 0, \pm 1, \pm 2, \ldots .$$

7. Find all possible values of the integral

$$\int_\gamma \frac{dz}{z^2+1}$$

for all closed curves γ which do not pass through $\pm i$.

Solution.

$$\frac{1}{z^2-1} = \frac{i}{2}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

So the integral around little positive oriented circle about i equals $2\pi i$ times $i/2$ that is $-pi$, and about similar circle about $-i$ it equals π. A closed curve can wind about i and $-i$ arbitrary number of times in positive or negative direction, thus the possible values of the integral are $k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$.