Holomorphic curves with bounds on the spherical derivative

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Let $f:\mathbf{C}\to\mathbf{P}^n$ be a holomorphic map; in homogeneous coordinates $f=(f_0:f_1:\ldots:f_n)$, where the f_j are entire functions with no zeros common to all of them. The spherical derivative $\|f'\|$ measures the length distortion from the Euclidean metric in $\mathbf{C}$ to the Fubini–Study metric in $\mathbf{P}^n$. Explicitly,

$$\|f'\|^2=\frac{\sum_{i<j}|f_i'f_j-f_if_j'|^2}{\left(\sum_j|f_j|^2\right)^2}.$$

We use the standard notation for the Nevanlinna characteristic $T(r,f)$ and Nevanlinna counting functions $N(r,a,f)$, for hyperplanes a in $\mathbf{P}^n$.

Conjecture. *If for some $\sigma>-1$ we have $\|f'\|(z)=O(z^\sigma)$, and $a_1,\ldots,a_q$ are hyperplanes in general position, and $f(\mathbf{C})\not\in\cup_{j=1}^q a_j$, then*

$$\sum_{j=1}^q N(r,a_j,f)\geq(q+1-n)T(r,f)+O(r^{\sigma+1}).$$

We recall than Cartan's theorem says that

$$\sum_{j=1}^q N(r,a_j,f)\geq(q-1-n)T(r,f)+o(T(r,f)),$$

when $r\to\infty$ avoiding a set of finite measure.

The Conjecture is known to be true in the following cases:

a) When $n=1$, when it becomes

$$N(r,a,f)=T(r,f)+O(r^{\sigma+1}),\quad r\to\infty,$$

for every $a \in \overline{\mathbf{C}}$.

b) When f omits $n-1$ hyperplanes in general position.

A weaker form of the Conjecture, with the error term $o(r^{2\sigma+2})$ is also known. All these results are due to J. Duval and B. da Costa, Sur les courbes de Brody dans $\mathbf{P}^n(\mathbf{C})$, Math. Ann. 355 (2013), no. 4, 1593–1600.