d'Alembert's solution of one-dimensional wave equation

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1. Solution on the line

Problem. To find the general solution of the one-dimensional wave equation on the whole line,

$$u_{tt} = c^2 u_{xx}. \tag{1}$$

Solution (due to d'Alembert). Let us introduce new independent variables:

$$\xi = x + ct, \quad \eta = x - ct,$$

so that

$$x = (\xi + \eta)/2, \quad t = (\xi - \eta)/(2c).$$

Then the new function is

$$w(\xi, \eta) = u(x, y) = u((\xi + \eta)/2, (\xi - \eta)/(2c)).$$

To obtain a differential equation for w, we differentiate with respect to ξ:

$$w_\xi = \frac{1}{2} u_x \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) + \frac{1}{2c} u_t \left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right),$$

and then differentiate with respect to η:

$$w_{\xi\eta} = \frac{1}{4} u_{xx} - \frac{1}{4c} u_{xt} + \frac{1}{4c} u_{tx} - \frac{1}{4c^2} u_{tt}.$$

Since $u_{xt} = u_{tx}$ the two middle terms cancel, and we obtain from (1) that

$$w_{\xi\eta} = 0. \tag{2}$$

This is easy to solve: w_ξ must be independent of η, so $w_\xi = f_1(\xi)$ for some function f_1, and integrating this with respect to ξ we obtain that $w(\xi,\eta) = f(\xi) + g(\eta)$, for some function f, such that $f' = f_1$, and some function g, the "constant of integration". Returning to our original variables we obtain

$$u(x,t) = f(x+ct) + g(x-ct), \tag{3}$$

Now it is easy to check directly that such a function u with *arbitrary* differentiable functions f and g satisfies equation (1). So we obtained a *general solution* which depends on two arbitrary functions.

Equation (1) describes oscillations of an infinite string, or a wave in 1-dimensional medium. To single out a unique solution we impose initial conditions at $t = 0$:

$$u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x), \tag{4}$$

that is we specify the initial position and initial velocity of the string.

To simplify our computation, we can use the *Superposition Principle*: first find a solution with arbitrary given ϕ and $\psi = 0$, then find a solution with $\phi = 0$ and arbitrary ψ, and then take the sum of these two solutions.

For the first solution we plug $t = 0$ into (3) and obtain

$$f + g = \phi, \quad f - g = 0,$$

so $f = g = \phi/2$, and the first solution, with zero initial velocity, is

$$u_1(x,t) = \frac{1}{2}\left(\phi(x+ct) + \psi(x-ct)\right). \tag{5}$$

For the second solution we differentiate (3) with respect to t and plug $t = 0$. We obtain

$$f + g = 0, \quad c(f' - g') = \psi.$$

Solving this we obtain the second solution

$$u_2(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy, \tag{6}$$

corresponding to zero initial position. Thus the complete solution u of the initial value problem (1), (4) is given by

$$u(x,t) = u_1(x,t) + u_2(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy.$$

This is the d'Alembert formula.

Let us give a geometric (or physical) interpretation of the first solution (5). Assume without loss of generality that $c > 0$. The graph of ϕ is the initial shape of the string. The change of the independent variable $x \mapsto x+ct$ corresponds to shifting this graph to the *left* by ct, so $(1/2)\phi(x+ct)$ represents a wave moving to the left with constant speed c. The amplitude of this wave is $1/2$ of the initial condition. Similarly, $(1/2)\phi(x-ct)$ represents the wave of the same shape moving to the right with the same speed c. The waves are of the same shape, and at $t = 0$ they combine to the initial shape $\phi(x)$.

Take some simple shape ϕ and draw pictures of solution, as a function of x for several moments of time t.

2. Solution on a finite interval.

Now we consider oscillation of a string of finite length. In this case, we need *boundary conditions*, for example the strings of string instruments (like violin or piano) are fixed at the ends. This corresponds to the boundary conditions

$$u(0,t) = u(L,t) = 0, \quad \text{for all} \quad t. \tag{7}$$

Let us consider for simplicity the initial condition

$$u(x,0) = \phi(x), \quad u'(x,0) = 0, \quad 0 \leq x \leq L. \tag{8}$$

This corresponds to pulling the string to the shape ϕ and then releasing it from rest. Can we use the solution that we obtained for the whole line (5) to obtain a solution on the interval $(0, L)$ with boundary condition (7)?

Of course, for the problem to be consistent, the initial shape ϕ must also satisfy the boundary condition, that is

$$\phi(0) = \phi(L) = 0. \tag{9}$$

Notice that ϕ is defined *only* on the interval $[0, L]$.

The idea is to *extend a function ϕ satisfying (9) from the interval* $[0, L]$ *to a function on the whole line, in such a way that the solution (5) will keep property (7) for all t.*

The clever way to do this is the following. First extend ϕ to an *odd* function on $[-L, L]$, that is define $\phi(x) = -\phi(-x)$ for $x \in [-L, 0]$. Then extend this odd function $2L$-periodically to the whole line, that is set $\phi(x + 2nL) = \phi(x)$ for $x \in [-L, L]$ and every integer n. The function we obtained will be odd on the whole real line, and will have period $2L$.

Why this extension is works? The *even* extension of a smooth function with $\phi(0) = 0$ will in general be non smooth at 0.

But the **main point** is that *every odd $2L$-periodic function is zero at the points* 0 *and* L!

So if $u(x,t)$ remains odd and $2L$-periodic with respect to x at all times t, then the boundary condition will be automatically satisfied.

Check that if ϕ is continuous and differentiable on $[0, L]$, and satisfies (9), then the proposed odd $2L$ periodic extension will be also continuous and differentiable.

Suppose now that ϕ is an odd $2L$ periodic function on the whole real line, and consider the function u defined by (5). It is also an odd $2L$-periodic function of x for each fixed t That it is $2L$ periodic is evident. To check that it is odd we write

$$\begin{aligned} u(-x,t) &= (1/2)\,(\phi(-x+ct) + \phi(-x-ct)) \\ &= -(1/2)\,(\phi(x+ct) + \phi(x-ct)) \\ &= -u(x,t), \end{aligned}$$

where we used that ϕ is odd.

Now it remains to notice that *any* odd $2L$-periodic function f satisfies $f(0) = 0$ and $f(L) = 0$.

Thus we obtain a unique solution of (1) on $[0, L]$ with boundary condition (7) and initial condition (9): *it is given by formula (5) where ϕ is the* **odd $2L$-periodic extension of the initial shape**.

It is very instructive to look what really happens in a series of pictures showing the shape of the string at various moments.

Let us take some very simple initial shape ϕ, say constant on a small interval near $L/4$ and zero outside a slightly bigger interval. *I urge you to make few pictures!*

Initially, for small times t, the original shape splits into two equal halves, and they start moving at speed c into opposite directions. But look carefully what happens when the bump reaches the left end $x = 0$. Since we made an odd $2L$-periodic extension, a *similar negative bump* comes to 0 from the negative side. In the next moments, these two bumps interfere and eventually may cancel each other, so at some moment the solution is zero near the end $x = 0$. And in the next moments the negative bump re-appears at the left end of the string and moves to the right. This is called "reflection of the wave" at the end. Same thing happens a bit later on the right end. Eventually the two negative reflected bumps will collide near the place where we had a positive bump at $t = 0$. Then they pass each other and this motion continues indefinitely.

Example. Suppose that $c = 1$ in (1), and we are solving (1) on the interval $[-1, 1]$ with the boundary conditions

$$u(-1, t) = u(1, t) = 0$$

and the initial conditions are

$$u(x, 0) = \phi := \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

and $u_t(x, t) = 0$.

Find $u(0, 2021)$ and $u(0, 2020)$.

Solution. By d'Alembert's formula, $u(x, t) = (\tilde{\phi}(x + t) + \tilde{\phi}(x - t))/2$, where $\tilde{\phi}$ is the 2-periodic extension of ϕ. So

$$u(0, 2021) = (\tilde{\phi}(-2021) + \tilde{\phi}(2021))/2 = (\phi(1) + \phi(-1))/2 = 0.$$

and similarly $u(0, 2020) = 1$.

Remarks. d'Alembert's formula explains why we can communicate using light or sound. Suppose that at the time $t = 0$ at the place $x = 0$ someone makes a click. We describe this click by some function $\phi(x)$ which is localized near $x = 0$: $\phi(x) = 0$ for $|x| > \epsilon$, where ϵ is a small number, like in the previous example. If you are sitting at some point $x = A > 0$, first you hear nothing. But at the time $t = A/c$ the wave $\phi(x - ct)$ moving with speed c to the right reaches you and passes. So you hear a click. It is important that the click arrives to the point A at time A/c *undistorted*. It is just 2-ce smaller that the original click.

So you can hear the original click, only with a time delay. Similarly if we are talking about electromagnetic waves, you can see a flash as a flash. A localized signal (a flash) arrives to your place with the speed of light undistorted.

All this applies to one space dimension while we are living in three dimensions. It turns out that in dimension 3 there is a formula for the solution of the wave equation which is very similar to d'Alembert's formula. It will be derived later in this course. So a click arrives to the listener as a click, and a flash arrives as a flash.

This is completely different in dimension 2. Think of waves on water surface. What happens when you throw a stone to a pond? The waves spread from the center in circles, and there are many of them! So an insect sitting on the water surface nearby will experience some oscillations for long time, instead of a short click. If the sound was spreading like this we would be unable to communicate using speech, every click will be perceived as a long and strong echo. Similarly we could not see anything.

It turns out that it is *parity* of the dimension of the space which is responsible for this difference. In spaces of odd dimension, the waves are localized, in even dimensions we have the so-called wave dispersion phenomenon which spreads them in time and space. All space around us will be filled with echo coming from all kinds of sources.

This phenomenon is called the *wave dispersion*, and it is present in spaces of all *even* dimensions.