D'Alembert and Fourier

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The natural question arises, how the solution of the one-dimensional wave equation obtained by Fourier's method is related to the solution that we earlier obtained by d'Alembert's method.

Consider the wave equation on an interval $0 \leq x \leq \pi$:

$$u_{tt} = c^2 u_{xx}, \quad 0 \leq x \leq \pi$$

with the boundary conditions

$$u(0) = u(\pi) = 0,$$

and initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = 0, \quad 0 \leq x \leq \pi. \tag{1}$$

d'Alembert's formula gives

$$u(x,t) = \frac{1}{2}\left(\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right), \tag{2}$$

where $\tilde{\phi}$ is the *odd 2π-periodic extension* of the initial shape ϕ.

Every real 2π-periodic function, subject to some smoothness conditions, can be represented by a Fourier series:

$$\tilde{\phi}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx). \tag{3}$$

The Fourier coefficients a_n, b_n are uniquely determined by the function.

When the function $\tilde{\phi}$ is odd, its Fourier expansion must be odd, so $a_n = 0$ for all $n \geq 0$, and we have the so-called sine Fourier series,

$$\tilde{\phi}(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Let us plug this expression to the d'Alembert formula (2), and use trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

to simplify the result. We obtain

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) \cos(nct), \tag{4}$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} \phi(x) \sin(nx) dx,$$

because $\tilde{\phi}(x) = \phi(x),\ 0 < x < L$. This is exactly the same formula that was obtained by Fourier method.

In general, let f be an arbitrary function defined on a symmetric interval $[a, b]$ (or on the whole real line). Then there is a *unique* decomposition

$$f = f_e + f_o \tag{5}$$

into the sum of an even function f_e and odd function f_o. Indeed, we can set $f_e(x) = (f(x) + f(-x))/2$ and $f_o(x) = (f(x) - f(-x))/2$. Then evidently f_e is even and f_o is odd, and their sum is f. If we had two such decompositions, say (5) and another one, $f = f_e^* + f_o^*$ then $f_e - f_e^* = f_o - f_o^*$, but a function which is simultaneously even and odd must be the zero function. So the second decomposition must be the same as the first one.

Formula (3) actually gives us this decomposition: the part with cosines is even and the part with sines is odd.

Let us consider some other boundary conditions, for example,

$$u_x(0,t) = 0, \quad u(\pi, t) = 0,$$

which means that the right end is fixed while the left end is free. (Imagine that there is a little ring attached to the string on the left end which is allowed to slide without friction on a vertical rod). Let us take the initial condition (1). Of course function ϕ must also satisfy $\phi'(0) = 0,\ \phi(L) = 0$.

The question is how to extend such a function to the whole real line, so that we can use d'Alembert's formula with this extension.

The answer is that we should extend it to become *even* on $[-L, L]$, then $\tilde{\phi}'(0) = 0$ will hold automatically. On the other hand the condition on the right end requires an odd extension about this end, that is $\tilde{\phi}(L + x) = -\tilde{\phi}(L - x)$, which is the same as $\tilde{\phi}(x) = -\tilde{\phi}(2L - x)$. These two properties will make our extension $4L$-periodic, indeed,

$$\tilde{\phi}(x) = \tilde{\phi}(-x) = -\tilde{\phi}(2L + x) = \tilde{\phi}(4L + x),$$

and the corresponding Fourier series will be

$$\tilde{\phi}(x) = \sum_{0}^{\infty} a_{2n} \cos(\pi n x/2L).$$

This is again consistent with what was obtained with Fourier's method.

The question may be asked why do we need a complicated Fourier series solution if we have a simple d'Alembert's solution.

Let us analyze what we really want to know about oscillations of a string, for example of a string of a musical instrument. We want to know how a given string will *sound*, first of all its pitch. The *pitch* depends on the frequency of oscillation. Our solution (4) shows that the oscillation is a superposition of oscillations with frequencies $nc/(2\pi)$. We assumed for simplicity that the length of the string is π. For a string of length L these frequencies will be $nc/(2L)$. It is an experimental fact that we perceive the *lowest present frequency* as the pitch. So it is $c/(2L)$; this is called the fundamental tone, and the rest of the frequencies are called *overtones.* Notice that for a string, they are integer multiples of the fundamental tone. To compute this fundamental frequency, we need to know c. Derivation of the wave equation (p. 388-389 of the book) shows that $c^2 = T/\rho$, where T is the force stretching the string, and ρ is the density. To we obtain that the fundamental frequency equals

$$\sqrt{\frac{T}{\rho}} \frac{1}{2L}.$$

This is called *Mersenne's Law.* It has an interesting history. That the fundamental frequency is inverse proportional to the length of a string was discovered in ancient Greece, ancient historians credit this discovery to Pythagoras himself. If this is so, this is probably the earliest discovery of a mathematical law of nature. The dependence on tension T and density ρ was discovered only in 17th century by a music theorist Marin Mersenne.

This is what concerns pitch. Another characteristic of a sound we hear is called *timbre*. (This is how we tell the sound of one musical instrument from the sound of another one, for example a violin and and piano sound differently, even when we play the same note). And it turns out that timbre depends on the relative size of the Fourier coefficients a_n, b_n! In other words, our ear and brain together work like a Fourier analyzer: they detect frequencies and amplitudes of harmonics present in the sound. For example the oscillations $a\cos t + b\cos 2t$ and $a\cos t + b\cos 2(t+c)$ are perceived by our ear as exactly the same,; we do not hear the dependence on c. Only frequencies 1 and 2 and amplicudes a, b are relevant for what we hear, but not the phase c.

This is an experimental law discovered by Georg Ohm, the same person who discovered an even more famous law of electric resistance. See "Ohm's acoustic law" on Wikipedia.

Interestingly, we perceive light in the same way: only frequencies and amplitudes are relevant. So our eye is also a harmonic analyzer.

That all higher frequencies are integer multiples of the fundamental tone is a property of a string. Later we will analyze oscillations of other shapes and will see that this is not always so. This explains why string musical instruments sound differently from other instruments.