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ITERATES OF ENTIRE FUNCTIONS

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A. È. EREMENKO AND M. YU. LYUBICH

This note presents a complete description of the asymptotic behavior of the iterates of entire functions of a certain class on a set of normality.

Let f be an entire function. We denote by $R(f)$ an open set in the plane, in which the family $\{f^m\}$ of iterates is normal. The complement $\mathbf{C}\backslash R(f)$ is called a *Julia set*. It is known [1] that $J(f)$ is a nonempty perfect completely invariant set (the last term means that $f^{-1}J(f) = J(f)$).

We study iterations of a special class of entire functions. This class S is defined as follows: $f \in S$ if there is a finite set $\{a_1, \dots, a_q\}$ of points such that

$$f: \mathbf{C}\backslash f^{-1}\{a_1, \dots, a_q\} \to \mathbf{C}\backslash\{a_1, \dots, a_q\}$$

is an (unramified) covering. A minimal set with this property is called a set of basis points. If f has q basis points, we say that $f \in S_q$. Examples: $\exp \in S_1$, $\sin \in S_2$. If P and Q are polynomials of degrees m and n then $\int_0^z P(\varsigma)\exp Q(\varsigma)\,d\varsigma \in S_{m+n}$. The class S is closed under superpositions. As we shall see below, iterations of functions of class S resemble iterations of rational functions.

THEOREM 1. *Let $f \in S$ be a transcendental function. If $z \in R(f)$ the orbit $\{f^m z\}$ cannot tend to ∞.*

This is not true for arbitrary functions, as shown by the example $f(z) = e^{-z} + z + 1$ which was studied by Fatou [1].

COROLLARY. *Every component of $R(f)$ for a transcendental function $f \in S$ is simply connected.*

This is not true for arbitrary entire functions, as was shown by Baker [2].

A component D of $R(f)$ is said to be *wandering* if $f^n D \cap f^m D = \varnothing$ for all $m > n > 0$. Sullivan [3] showed that there are no wandering components for rational functions. The following theorem can be proved by the method of [3].

THEOREM 2. *An entire function f of class S has no wandering components in the set $R(f)$.*

The first example of an entire function with a wandering component was constructed by Baker [4]. In this example the wandering component D is multiply connected, the functions f^m are multivalent in D, and $f^m D \to \infty$ as $m \to \infty$. The following theorems provide new examples of wandering components.

THEOREM 3. *There is an entire function f with a wandering component of $R(f)$, such that D is simply connected and f^m is univalent in D for all $m \geq 1$.*

THEOREM 4. *There is an entire function f with a wandering component D of $R(f)$, such that the orbit $\{f^m D\}$ has an infinite set of limit points.*

It is not known whether the orbit of a wandering domain can be bounded.

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Theorem 2 makes it possible to give a complete description of the orbits on the set $R(f)$. It follows from this theorem that for every component D of $R(f)$, $f \in S$, there is an m such that $f^m D$ is a periodic component. We now consider a periodic component D; let p be its order (i.e., p is the smallest number such that $f^p D = D$). The set

$$\bigcup_{j=1}^{p-1} f^j D \tag{1}$$

is called a *cycle of components.* Such a cycle is called a *Fatou cycle* if $\{f^{np}z\}$ converges as $n \to \infty$ to a fixed point $a \in \overline{D}$ of f^p for every $z \in D$. A cycle (1) of components of $R(f)$ is called a *Siegel cycle* if there is a univalent conformal mapping φ of D onto the unit disk satisfying the Schroeder equation $\varphi \circ f^p = \lambda\varphi$, where $\lambda = \exp 2\pi i\alpha$ with α irrational. The next theorem follows from Theorems 1 and 2 and classical results of Wolff and Denjoy.

THEOREM 5. *Let $f \in S$, and let D be a component of $R(f)$. Then there exists $m \geq 1$ such that $f^m D$ belongs to a Fatou or Siegel cycle.*

It follows from a theorem of Fatou [5] that every Fatou cycle contains at least one basis point of the function $f \in S$. Hence the number of Fatou cycles is finite. We can show, by using an idea of Fatou, that the number of Siegel cycles is also finite.

THEOREM 6. *If $f \in S_q$, then the number of Fatou cycles does not exceed q, and the number of Siegel cycles does not exceed $2q$.*

For entire functions that do not belong to class S, there are cycles of components that are neither Fatou nor Siegel cycles, as Fatou's example, cited above, shows. Moreover, f can be univalent on such components.

THEOREM 7. *There is an entire function f that has an invariant component of $R(f)$, with f univalent in D and $f^m z \to \infty$ as $m \to \infty$, uniformly on compact subsets of D.*

The following theorem is an application of the preceding results.

THEOREM 8. *Let $f \in S$ be a transcendental function. If D is a completely invariant component of $R(f)$, then $D = R(f)$.*

This proposition was stated by Baker [6] as a conjecture for arbitrary entire functions.

Let f be an entire function. One says that f *has a (finite) asymptotic value* if there is a curve Γ tending to ∞ such that $\lim f(z)$ exists and is finite for $z \in \Gamma$, $z \to \infty$. The function f has finite order if

$$\log\log \max\{|f(z)|: |z| = r\} = O(\log r), \qquad r \to \infty.$$

An example of a function $f \in S$ of finite order with an asymptotic value is $f(z) = P(\exp z)$, P a polynomial.

We say that the orbit of the point $b \in \mathbf{C}$ is *absorbed by the cycle* $\{f^j a\}_{j=0}^{p-1}$ if $f^m b = a$ for some $m \geq 1$.

THEOREM 9 (cf. [7]). *Let $f \in S$ be a function of finite order with an asymptotic value. Assume that the orbits of all basis points that belong to $J(f)$ are absorbed by repulsive cycles. Then either $J(f) = \mathbf{C}$ or the planar measure of $J(f)$ is 0.*

We remark that it is unknown up to now whether the sets $J(f)$ for polynomials are of measure 0.

The preceding results can be applied to one-parameter families of functions $f_c(z) = \exp z + c$, $c \in \mathbf{C}$. For real c we have the following theorem.

THEOREM 10. *If* $c > -1$, *then* $J(f_c) = \mathbf{C}$. *However, if* $c \leq -1$, *then the set* $R(f_c)$ *is connected, and the measure of* $J(f_c)$ *is* 0.

For $c = 0$ this result, stated by Fatou as a conjecture, was proved by Misiurewitz [5] by elementary methods. We note that the first examples of entire functions f for which $J(f) = \mathbf{C}$ were constructed by Baker [9]. The bifurcation diagram of the family f_c resembles the diagram for the known family $z^2 + c$ [10], but the domains of stability are unbounded.

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