# Meromorphic solutions of Briot–Bouquet type differential equations

A. Eremenko

March 2011

Let $F$ be a polynomial in two variables. Assume that the differential equation

$$F(y^{(k)}, y) = 0 \tag{1}$$

has a solution $y(z)$ which is meromorphic in the whole plane.
What can be said about $y$?

For $k = 1$, a complete answer is well known. Namely, all meromorphic solutions belong to the class $W$, which consists by definition of: all elliptic functions, all functions of the form $R(\exp(az))$, $R$ rational and $a \neq 0$ is a complex number, and all rational functions. And vice versa, each function of the class $W$ satisfies some differential equation of the above form with $k = 1$.

**Conjecture.** *All meromorphic solutions of (1) either belong to $W$ or satisfy a linear equation of the form*

$$y^{(k)} + ay + b = 0.$$

For $k = 2$ Emile Picard proved in 1880 that all meromorphic solutions also belong to the class $W$. This result was forgotten, in 1978 E. Hille stated it as a conjecture, and in 1981 S. Bank and R. Kaufman gave a proof, more complicated than the one Picard published 101 years before.

It follows from another theorem of Picard that non-constant meromorphic solutions are possible only if the genus of the curve $F(x, y) = 0$ is at most 1.

In the case when the genus is equal to 1 I proved for every $k$, that all meromorphic solutions must be elliptic functions.

Thus only the case of genus 0 remains. I also proved the following. If $k$ is odd and a solution $y$ is meromorphic in the plane and has at least one pole, it must belong to $W$.

The simplest equation not covered by these results is

$$y^{(IV)} = 24y^5.$$

It evidently has solutions of the form $1/(z-b)$. Does it have any other meromorphic solutions?

*Added on May 22, 2007.* L. W. Liao, T. W. Ng and the present author recently proved that for every $k$, all meromorphic solutions of $F(y^{(k)}, y) = 0$. having at least one pole belong to $W$.

Eremenko, Alexandre E.; Liao, Liangwen; Ng, Tuen Wai, Meromorphic solutions of higher order Briot–Bouquet differential equations. Math. Proc. Cambridge Philos. Soc. 146 (2009), no. 1, 197–206.

This proof also implies that the only meromorphic functions that satisfy $y^{(k)} = y^m$ with $m > 1$ are of the form $c(z-b)^{-n}$.

The question of description of entire solutions remains open. Notice that non-trivial entire solutions of $y''' = y$ do not belong to $W$. So it is not completely clear what is the right conjecture about entire solutions. Perhaps they can be only exponential polynomials?

*Added on June 19, 2023.* Further progress was achieved in the paper:

A. Ya. Yanchenko, One advance in the proof of the conjecture on meromorphic solutions of Briot-Bouquet type equations. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 86 (2022), no. 5, 197–208; translation in Izv. Math. 86 (2022), no. 5, 1020–1030.

This author proves that if $F$ is an irreducible polynomial of degree $d \geq 2$, and the top degree homogeneous part of $F$ has $d$ distinct linear factors, then all entire solutions are Laurent polynomials of $e^{az}$ with some complex $a$.

*Added on December 14, 2024* Actually the results proved in [2] imply a stronger statement.

*Suppose that (1) has an entire solution, and $F$ is an irreducible polynomial of degree $d$. Then the top degree homogeneous part $F_d$ of $F$ has one or two* *distinct linear factors. In the case of two factors, every entire solution is a Laurent polynomial of* $e^{az}$.

Thus the question remains open only for equations of the form

$$(y^{(k)} - ay)^d + Q_{d-1}(y^{(k)}, y) = 0,$$

where $\deg Q_{d-1} \leq d - 1$.

# References


[1] S. Bank and R. Kaufman, On Briot-Bouquet differential equations and a question of Einar Hille. Math. Z. 177 (1981), no. 4, 549–559.

[2] A. Eremenko, Meromorphic solutions of equations of Briot–Bouquet type, Teor. Funktsii, Funk. Anal. i Prilozh., 38 (1982) 48–56. English translation: Amer. Math. Soc. Transl. (2) Vol. 133 (1986) 15–23.

[3] E. Picard, Sur une propriété des fonctions uniformes dune variable et sur une classe déquations différentielles, C. R. Acad. Sci. Paris, 91 (1880) 1058–1061.