An extremal problem for a class of entire functions

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Abstract

Soi f une fonction entière de type exponentielle donc le diagramme indicatrice est contenu dans l'interval $[-i\sigma, i\sigma], \sigma > 0$. Alors la densité superieure de zéros de f ne dépasse pas $c\sigma$ ou $c \approx 1.508879$ est la solution d'équation

$$\log(\sqrt{c^2+1}+c) = \sqrt{1+c^{-2}}.$$

Cette borne est exacte.

We consider the class $E_\sigma,\ \sigma > 0$ of entire functions of exponential type whose indicator diagram is contained in a segment $[-i\sigma, i\sigma]$, which means that

$$h(\theta) := \limsup_{r\to+\infty} \frac{\log|f(re^{i\theta})|}{r} \leq \sigma|\sin\theta|, \quad |\theta| \leq \pi. \tag{1}$$

An alternative characterization of such functions follows from a theorem of Pólya [6]:

$$f(z) = \frac{1}{2\pi}\int_\gamma F(\zeta)e^{-i\zeta z}d\zeta,$$

where F is an analytic function in $\overline{\mathbf{C}}\backslash[-\sigma,\sigma],\ F(\infty) = 0$, and γ is a closed contour going once around the segment $[-\sigma, \sigma]$. In other words, the class of

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entire functions satisfying (1) consists of Fourier transforms of hyperfunctions supported by $[-\sigma, \sigma]$, see, for example, [2] and [3].

Let $n(r)$ be the number of zeros of f in the disc $\{z : |z| \leq r\}$, counting multiplicity. We are interested in the *upper density*

$$D = \limsup_{r \to \infty} \frac{n(r)}{r}. \tag{2}$$

If f satisfies the additional condition

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty, \tag{3}$$

then the limit (density) in (2) exists and equals $(2\pi)^{-1} \int_{-\pi}^{\pi} h(\theta) d\theta$. For example, if $f(z) = \sin \sigma z$, then $f \in E_\sigma$ and $D = 2\sigma/\pi \approx 0.6366\sigma$ The existence of the limit follows from a theorem of Levinson [5, 6]. Much more precise information about $n(r)$ under the condition (3) is contained in the theorem of Beurling and Malliavin [1].

In the general case, the density might not exist as was shown by examples in [4, 10]. Moreover, it is possible that $D > 2\sigma/\pi$, see [2]. An easy estimate using Jensen's formula gives $D \leq 2e\sigma/\pi \approx 1.7305\sigma$. This estimate is exact in the larger class of entire functions satisfying the condition $h(\theta) \leq \sigma$, but it is not exact in E_σ.

In this paper we find the best possible upper estimate for the upper density of zeros of functions in E_σ.

Theorem. *The upper density of zeros of a function $f \in E_\sigma$ does not exceed $c\sigma$ where $c \approx 1.508879$ is the unique solution of the equation*

$$\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}, \quad \textit{on} \quad (0, +\infty). \tag{4}$$

For every $\sigma > 0$ there exist entire functions $f \in E_\sigma$ such that $D = c\sigma$.

Proof. Without loss of generality we assume that $\sigma = 1$. Moreover, it is enough to consider only even functions. To make a function f even we replace it by $f(z)f(-z)$, which results in multiplication of both the indicator h and the upper density D by the same factor of 2.

Let $t_n \to +\infty$ be such sequence that $\lim n(t_n)/t_n = D$. Consider the sequence of subharmonic functions $v_n(z) = t_n^{-1} \log |f(t_n z)|$. Compactness

Principle for subharmonic functions [3, Theorem 4.1.9] implies that one can choose a subsequence that converges in $\mathscr{D}'$ (Schwartz's distributions). The limit function v is subharmonic in the plane, and satisfies

$$v(z) \leq |\mathrm{Im}\ z|, \quad z \in \mathbf{C}, \quad \text{and} \quad v(0) = 0. \tag{5}$$

Let μ be the Riesz measure of this function. We have to show that

$$\mu(\{z : |z| \leq 1\}) \leq c. \tag{6}$$

First we reduce the problem to the case that the Riesz measure μ is supported by the real line. We have

$$v(z) = \frac{1}{2} \int \log \left| 1 - \frac{z^2}{\zeta^2} \right| d\mu_\zeta.$$

Let us compare this with

$$v^*(z) = \frac{1}{2} \int_0^\infty \log \left| 1 - \frac{z^2}{t^2} \right| d\mu^*_t,$$

where μ^* is the radial projection of the measure μ: it is supported on $[0, +\infty)$ and $\mu^*(a, b) = \mu(\{z : a < |z| < b\}),\ 0 \leq a < b$. It is easy to see that

$$v^*(z) \leq \sigma' |\mathrm{Im}\ z|, \quad z \in \mathbf{C}, \quad \text{and} \quad v^*(0) = 0 \tag{7}$$

with some $\sigma' > 0$. We claim that one can choose $\sigma' \leq 1$ in (7). Let σ' be the smallest number for which (7) holds. Then, by the subharmonic version of the theorem of Levinson mentioned above (see, for example, [9]), the limit

$$\lim_{r \to \infty} r^{-1} v^*(rz) = \sigma' |\mathrm{Im}\ z|$$

exists in $\mathscr{D}'$ and thus

$$\lim_{r \to \infty} \frac{1}{r} \int_0^r \frac{n_{v^*}(t)}{t} dt = \lim_{r \to \infty} \frac{1}{2\pi r} \int_{-\pi}^{\pi} v^*(re^{i\theta}) d\theta = 2\sigma'/\pi,$$

where

$$n_{v^*}(r) = \mu^*[0, r] = \mu\{z : |z| \leq r\}. \tag{8}$$

Similar limits exist for v, and we have $n_v = n_{v^*}$, from which we conclude that $\sigma' \leq 1$.

From now on we assume that v is harmonic in the upper and lower half-planes, and that

$$v(iy) \sim y, \quad y \to +\infty. \tag{9}$$

Let u be the harmonic function in the upper half-plane such that $\phi = u + iv$ is analytic, and $\phi(0) = 0$. Then ϕ is a conformal map of the upper half-plane onto some region G of the form

$$G = \{x + iy : y > g(x)\}, \tag{10}$$

where g is an even upper semi-continuous function, $g(0) = 0$. Moreover,

$$\phi(iy) \sim iy, \quad \text{as} \quad y \to +\infty, \tag{11}$$

which follows from (9), and

$$\phi(-\overline{z}) = -\overline{\phi(z)}, \tag{12}$$

because both the region G and the normalization of ϕ are symmetric with respect to the imaginary axis. Finally we have

$$\mu([0, x]) = \frac{2}{\pi} u(x). \tag{13}$$

For all these facts we refer to [7].

Remark. The function Re $\phi(x) = u(x)$ might be discontinuous for $x \in \mathbf{R}$. We agree to understand $u(x)$ as the limit from the right $u(x+0)$ which always exists since u is increasing.

Inequality (5) implies that $v(x) \leq 0$, thus $g(x) \leq 0$, in other words, G contains the upper half-plane.

Thus we obtain the following extremal problem: *Among all univalent analytic functions ϕ satisfying (12) and mapping the upper half-plane onto regions of the form (10) with $g \leq 0$, $g(0) = 0$ and satisfying $\phi(0) = 0$ and (11), maximize* Re $\phi(1)$.

We claim that the extremal function g for this problem is

$$g_0(x) = \begin{cases} -\infty, & 0 < |x| < \pi c/2, \\ 0, & \text{otherwise}, \end{cases}$$

where $c > 1$ is the solution of equation (4). The corresponding region is shown in Fig. 1. For the extremal function we have $\phi_0(1) = \pi c/2 - i\infty$.

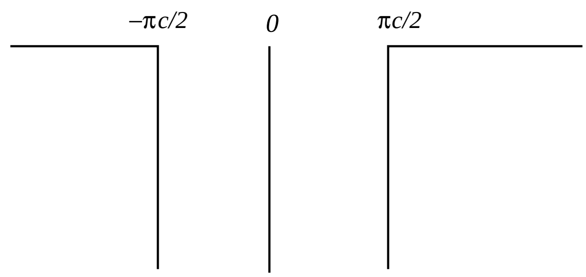

Fig. 1. Extremal region.

To prove the claim, we fist notice that for a given G the mapping function is uniquely defined. Let $a = \phi(1)$, and $b = \text{Re}\, a$. Next we show that making g smaller on the interval $(0, b)$ results in increasing $\text{Re}\, \phi(1)$ and making g larger on the interval $(b, +\infty)$ also results in increasing $\text{Re}\, \phi(1)$. The proofs of both statements are similar. Suppose that $g_1 \leq g$, $g_1 \neq g$, and $g_1(x) = g(x)$ outside of the two intervals $p < |x| < q$, where $0 < p < q < b$. Let G_1 be the region above the graph of g_1, and ϕ_1 the corresponding mapping function normalized in the same way as g. Then $G \subset G_1$, and the conformal map $\phi_1^{-1} \circ \phi$ is defined in the upper half-plane and maps it into itself. We have

$$\phi_1^{-1} \circ \phi(x) = x + 2x \int_0^\infty \frac{w(t)}{t^2 - x^2} dt,$$

where $w \neq 0$ is a non-negative function supported on some interval inside $(0, 1)$. Putting $x = 1$ we obtain

$$\phi_1^{-1}(a) = 1 + 2 \int_0^\infty \frac{w(t)}{t^2 - 1} dt,$$

so $\phi_1^{-1}(a) < 1$, that is $\text{Re}\, \phi_1(1) > b$. This proves our claim.

It remains to compute the constant b in the extremal domain. We recall that $\phi_0(1) = b - i\infty$ and assume that $b = \phi_0(k)$ for some $k > 1$. Here ϕ_0 is the extremal mapping function. Then by the Schwarz–Christoffel formula we have

$$\phi_0(z) = \frac{1}{2} \int_0^{z^2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} d\zeta. \tag{14}$$

To find k, we use the condition that

$$\text{Im } p.v. \int_0^{k^2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} d\zeta = 0.$$

Denoting $c = \sqrt{k^2 - 1}$ and evaluating the integral, we obtain

$$\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}.$$

Finally the jump of the real part of the integral in (14) occurs at the point 1 and has magnitude $\pi\sqrt{k^2 - 1} = \pi c$. This completes the proof of the upper estimate in Theorem 1.

To construct an example showing that this estimate can be attained, we follow the construction in [2, Sect.9-10]. The role of the subharmonic function u_1 there is played now by our extremal function $v_0 = \text{Im } \phi_0$.

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